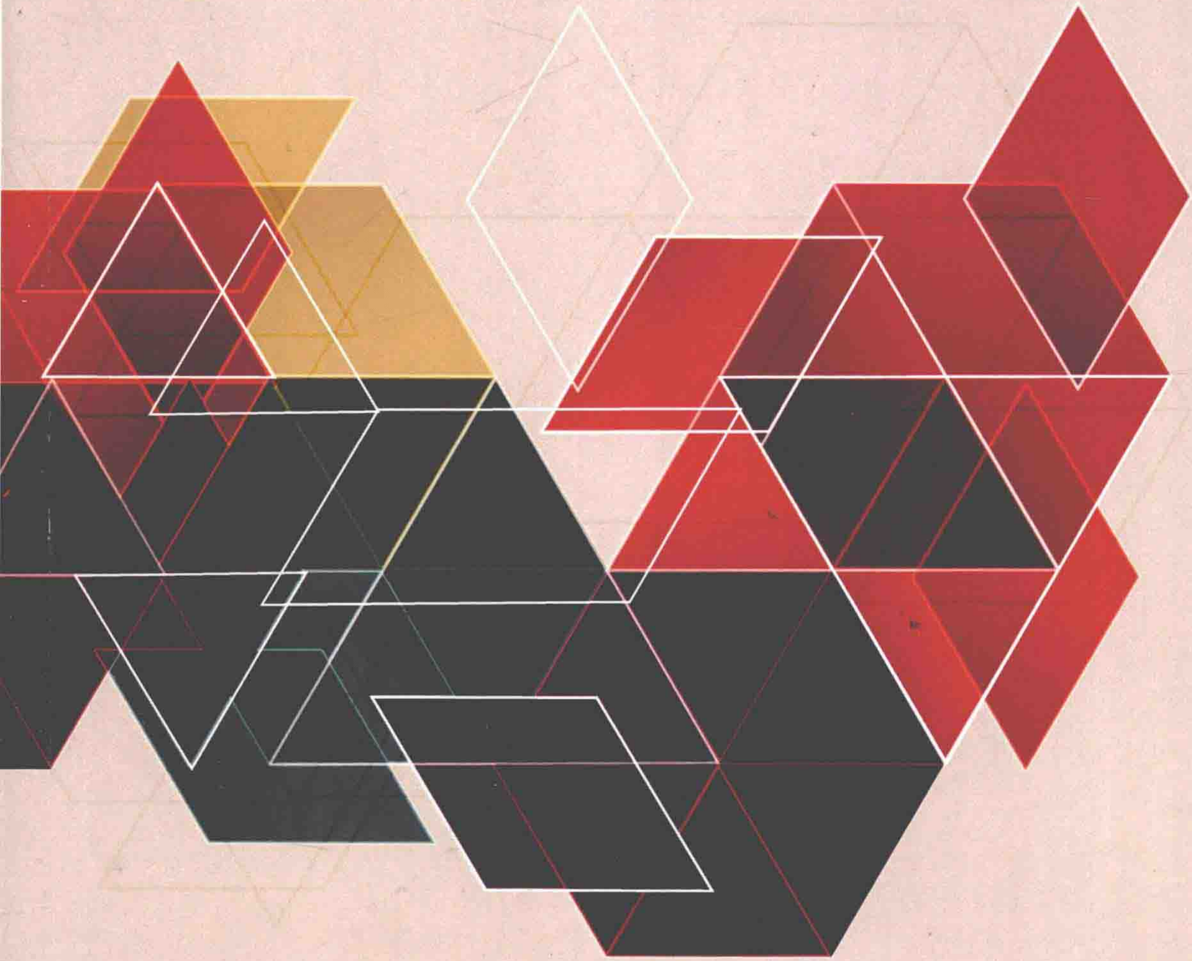
LINEAR ALGEBRA

Pure & Applied



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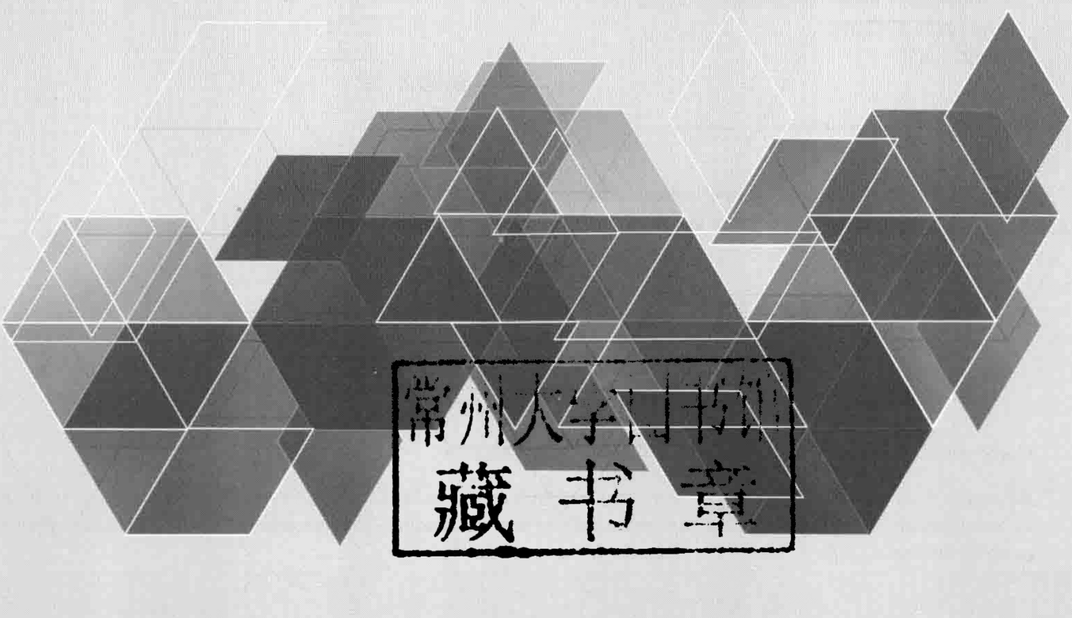

Edgar G. Goodaire
Memorial University, Canada

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Preface

This book had its genesis in the last century and has been used at Memorial University in St. John's, Newfoundland in one form or another for at least fifteen years. I am grateful to World Scientific Publishing Co. for making the latest edition widely available.

At many universities, many students take a course in linear algebra which should more properly be called "matrix algebra" because the content is primarily matrix manipulation and solving linear systems. Like calculus, the first course in linear algebra is often a course without proofs. To be successful, a student simply matches examples to homework with almost no reading and little real understanding.

My goal here is to use an intrinsically interesting subject to show students that there is more to mathematics than numbers, and that reading to understand while challenging, can also be very satisfying. A secondary purpose is to raise the level of the typical Linear Algebra I course so that the transition to Linear Algebra II is not as difficult as it has become.

The presentation in this text is frequently interrupted by "Reading Checks," little questions a student can use to see if she really does understand the point just made and which are intended to promote active reading. Every section in this book concludes with answers to these Reading Checks followed by some "True/False" questions that test definitions and concepts, not dexterity with numbers or buttons. Every exercise set contains some "Critical Reading" exercises which include problems some of which are harder, yes, but also some that can be solved quite easily **if understood**. Each chapter concludes with review exercises and a list of key words and concepts that have been introduced in that chapter.

There are several reasons for including "pure and applied" in the title. While a pure mathematician myself and a person who, as a student, loved the abstract vector space/linear transformation approach to linear algebra, I think such treatment is less successful today because it requires more maturity than the typical sophomore possesses. This book is mainly about Euclidean n-space and matrices. Essentially every vector space is a subspace of R^n. Other standard examples, like polynomials and matrices, are mentioned, but not emphasized. In my opinion, when students are thoroughly comfortable with R^n, they have little trouble transferring concepts to general vector spaces. Moreover, most theorems in elementary linear algebra have proofs that are independent of the vector space, so why not let the student remain in his comfort zone. The emphasis on matrices and matrix factorizations, the recurring themes of projections and codes, and shorter sections on subjects like facial recognition, Markov chains, graphs and electric circuits give this book a genuinely applied flavour.

This is a book which emphasizes "pure" mathematics, understanding and rigour. Virtually nothing is stated without justification, and a large number of exercises are of the "show or prove" variety. Since students tend to find such exercises difficult, I have included an appendix called "Show and Prove" designed to help students write a mathematical proof, and another, "Things I Must Remember," which has essentially been written by my students over the years. So despite its emphasis on the concrete, this book would also serve well to support the "introduction to proof" transition course that is now a part of many degree programs.

Organization, Philosophy, Style

Many linear algebra texts begin with linear equations and matrices, which students enjoy, but which suggest incorrectly that linear algebra is about computation. This book begins with vectors, primarily in R^2 and R^3, so that I can introduce in a concrete setting some key concepts—linear combination, span, linear dependence and independence—long before the "invariance of dimension" theorem, which is often the point where students decide that linear algebra is hard. The idea of a plane being "spanned" by two vectors is not hard for beginning students; neither is the idea of "linear combination" or the fact that linear dependence of three or more vectors means that the vectors all lie in a plane. That $A\mathsf{x}$ is a linear combination of the columns of A, surely one of the most useful ideas of linear algebra, is introduced early in Chapter 2 and used time and time again. Long before we introduce abstract vector spaces, all the terminology and techniques of proof have been at play for some time. A solid early treatment of linear independence and spanning sets in the concrete setting of R^n alerts students to the fact that this course is not about computation; indeed, serious thought and critical reading will be required to succeed. Many exercise sets contain questions that can be answered very briefly, if understood. There are some writing exercises too, marked by the symbol ✍, that ask for brief explanations or, sometimes, biographical information.

In some respects, this book is at a higher level than its competitors. The theory behind Markov chains many students will find difficult. Topics like the pseudoinverse and singular value decomposition and even complex matrices and the spectral theorem are tough for most second year students. On the other hand, versions of this book have been well received and proven to work for students with average to good mathematics backgrounds.

Technology goes almost unmentioned in this book. While a great fan of technology myself, and after delivering courses via MAPLE worksheets, for instance, I have formed the opinion that teaching students how to use sophisticated calculators or computer algebra packages is more time-consuming and distracting than useful.

I try to introduce new concepts only when they are needed, not just for their own sake. Linear systems are solved by transforming to row echelon form, not **reduced** row echelon form, the latter idea introduced where it is of more benefit, in the calculation of the inverse of a matrix. The notion of the "transpose" of a matrix is introduced in the section on matrix multiplication because it is important for students to know that the **dot** product $\mathsf{u} \cdot \mathsf{v}$ of two column vectors is the **matrix** product $\mathsf{u}^T\mathsf{v}$. Symmetric matrices, whose LDU factorizations are so easy, appear for the first time in the section on LDU factorization, and reappear as part of the characterization of a projection matrix.

There can be few aspects of linear algebra more useful, practical, and interesting than eigenvectors, eigenvalues and diagonalizability. Moreover, these topics provide an excellent opportunity to discuss linear independence in a nonthreatening manner, so these ideas appear early and in a more serious vein later.

Many readers and reviewers have commented favourably on my writing which, while seemingly less formal than that of other authors is most assuredly not lacking in rigour. There is far less emphasis on computation and far more on mathematical reasoning in this book. I repeatedly ask students to explain "why." Already in Chapter 1, students are asked to show that two vectors are linearly dependent if and only if one is a scalar multiple of another. The concept of matrix inverse appears early, as well as its utility in solving matrix equations, several sections before we discuss how actually to find the inverse of a matrix. The fact that students find the first chapter quite difficult is evidence to me that I have succeeded in emphasizing the importance of asking "Why", discovering "Why," and then clearly communicating the reason "Why."

A Course Outline

To achieve economy within the first three chapters, I omit Section 2.8 on LDU factorization (but never the section on LU) and discuss only a few of the properties of determinants (Section 3.2), most of which are used primarily to assist in finding determinants, a task few people accomplish by hand any more.

Linear Algebra II is essentially Chapters 4 through 7, but there is more material here than I can ever manage in a 36-lecture semester. Thus I sometimes discuss the matrix of a linear transformation only with respect to standard bases, omitting Sections 5.4 and 5.5. The material of Section 6.1, which is centred around the best least squares solution to over-determined linear systems, may be nontraditional, but try to resist the temptation to omit it. Many exercises on this topic have numerical answers (which students like!), there are lots of calculations, but lots of theory to be reinforced too. For example, the fact that the formula $P = A(A^TA)^{-1}A^T$ works only for matrices with linearly independent columns provides another opportunity to talk about linear independence.

To do proper justice to Chapter 7—especially the unitary diagonalization of Hermitian matrices and some (remedial?) work on complex numbers and matrices—I have to cut Sections 6.4 and 6.5 on orthogonal subspaces and the pseudoinverse. When I include Section 6.5 on the pseudoinverse (a nonstandard topic that students like), I bypass the first two sections of Chapter 7 and head directly to the orthogonal diagonalization of real symmetric matrices in Section 7.3 after a brief review of the concepts of eigenvalue and eigenvector.

Acknowledgements

Over the years, I have been helped and encouraged by a number of colleagues, including Ivan Booth, Peter Booth, Hermann Brunner, John Burry, Clayton Halfyard, Mikhail Kotchetov, George Miminis, Michael Parmenter and Donald Rideout. In particular, Misha Kotchetov and my friend of many years, Michael Parmenter, one of the best proof readers I have ever known, made numerous suggestions that improved this work immeasurably.

Many students have also helped me to improve this book and to make the subject easier for those that follow. In particular, I want to acknowledge the enthusiasm and assistance of Gerrard Barrington, Shauna Gammon, Ian Gillespie, Philip Johnson and Melanie Ryan.

I hope you discover that this book provides a refreshing approach to an old familiar topic with lots of "neat ideas" that you perhaps have not noticed or fully appreciated previously. If you have adopted this book at your institution for one of your courses, I am very pleased, but also most genuine in my request for your opinions, comments and suggestions.

Edgar G. Goodaire
edgar@mun.ca
St. John's, Newfoundland, Canada

To My Students

From aviation to the design of cellular phone networks, from data compression (CDs and jpegs) to data mining and oil and gas exploration, from computer graphics to Google, linear algebra is everywhere and indispensable. With relatively little emphasis on sets and functions, linear algebra is "different." Most students find it enjoyable and a welcome change from calculus. Be careful though. The answers to most calculus problems are numerical and easily confirmed with a text message. The answers to many problems in this book are **not** numerical, however, and require explanations as to **why** things are as they are. So let me begin this note to you with a word of caution: this is a book you **must read**.

For many of you, linear algebra will be the first course where many exercises ask you to explain, to answer why or how. It can be a shock to discover that there are mathematics courses (in fact, most of those above first year) where words are more important than numbers. Gone are the days when the solution to a homework problem lies in finding an identical worked example in the text. Homework is now going to require some critical thinking!

Many years ago, when this book existed just as course notes, a student came into my office one day to ask for help with a homework question. When this happens with a book of which I am an author, I am always eager to discover whether or not I have laid the proper groundwork so that the average student could be expected to make a reasonable attempt at an exercise. From your point of view, a homework exercise should be very similar to a worked example. Right? In the instance I am recalling, I went through the section with my student page by page until we found such an example. In fact, we found precisely the question I had assigned, with the complete solution laid out as an example that I had forgotten to delete when I moved it to the exercises! The student felt a little sheepish while I was completely shocked to be reminded, once again, that some students don't read their textbooks.

It is always tempting to start a homework problem right away, without preparing yourself first, but this approach isn't going to work very well here. You will find it imperative to read a section from start to finish before attempting the exercises at the end of a section. And please do more than just glance at the list of "key words" that appears at the end of each chapter. Look carefully at each word. Are you sure you know what it means? Can you produce an example or a sentence that would convince your teacher that you understand what the word means? If you can't, it's for sure you won't be able to answer a question where that word is used. Go to the back of the book, to a glossary where every technical term is defined and accompanied by examples. If you are not sure what is required when asked to prove something or to show that something is true, read the appendix "Show and Prove" that is also at the back. (You will find there the solutions to several exercises from the text itself!)

In another appendix, entitled "Things I Must Remember," I have included many important ideas that my students have helped me to collect over the years. You will also find there some ideas that are often just what you need to solve a homework problem.

I hope that you like my writing style, that you discover you like linear algebra, and that you soon surprise yourself with your ability to write a good clear mathematical proof. I hope that you do well in your linear algebra courses and all those other courses where linear algebra plays an important role. Let me know what you think of this book. I like receiving comments—good, bad and ugly—from anyone.

Edgar G. Goodaire
edgar@mun.ca
St. John's, Newfoundland, Canada

Suggested Lecture Schedule

1	**The Geometry of the Plane and 3-space**	**11 Lectures**
1.1	Vectors and Arrows	2
1.2	Length and Direction	2
1.3	Lines, Planes and the Cross Product	4
1.4	Projections	2
1.5	Linear Dependence and Independence	1
2	**Matrices and Linear Equations**	**18 Lectures**
2.1	The Algebra of Matrices	$2\frac{1}{2}$
2.2	Application: Generating Codes with Matrices	1
2.3	Inverse and Transpose	$2\frac{1}{2}$
2.4	Systems of Linear Equations	2
2.5	Application: Electric Circuits	1
2.6	Homogeneous Systems; Linear Independence	2
2.7	Elementary Matrices and LU Factorization	3
2.8	LDU Factorization	2
2.9	More on the Inverse of a Matrix	2
3	**Determinants, Eigenvalues, Eigenvectors**	**11 Lectures**
3.1	The Determinant of a Matrix	2
3.2	Properties of Determinants	3
3.3	Application: Graphs	1
3.4	Eigenvalues and Eigenvectors	$1\frac{1}{2}$
3.5	Similarity and Diagonalization	$1\frac{1}{2}$
3.6	Application: Linear Recurrence Relations	1
3.7	Application: Markov Chains	1

Contents

Chapter 1

Euclidean n-Space

1.1 Vectors and Arrows

A *two-dimensional vector* is a pair of numbers written one above the other in a column and enclosed in brackets. For example,

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

are two-dimensional vectors. Different people use different notation for vectors. Some people underline, others use boldface type and still others arrows. Thus, in various contexts, you may well see

$$\underline{v}, \quad \mathsf{v} \quad \text{and} \quad \vec{v}$$

as notation for a vector. In this book, we will use boldface, the second form, but in handwriting the author prefers to underline.

The *components* of the vector $\mathsf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ are the numbers a and b. By general agreement, vectors are *equal* if and only if they have the same first component and the same second component. Thus, if

$$\begin{bmatrix} a-3 \\ 2b \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix},$$

then $a - 3 = -1$ and $2b = 6$, so $a = 2$ and $b = 3$. The vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is often pictured by an arrow in the plane.

1.1.1

> Take any point $A(x_0, y_0)$ as the tail and $B(x_0 + a, y_0 + b)$ as the head of an arrow. This arrow, from A to B, is a picture of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.