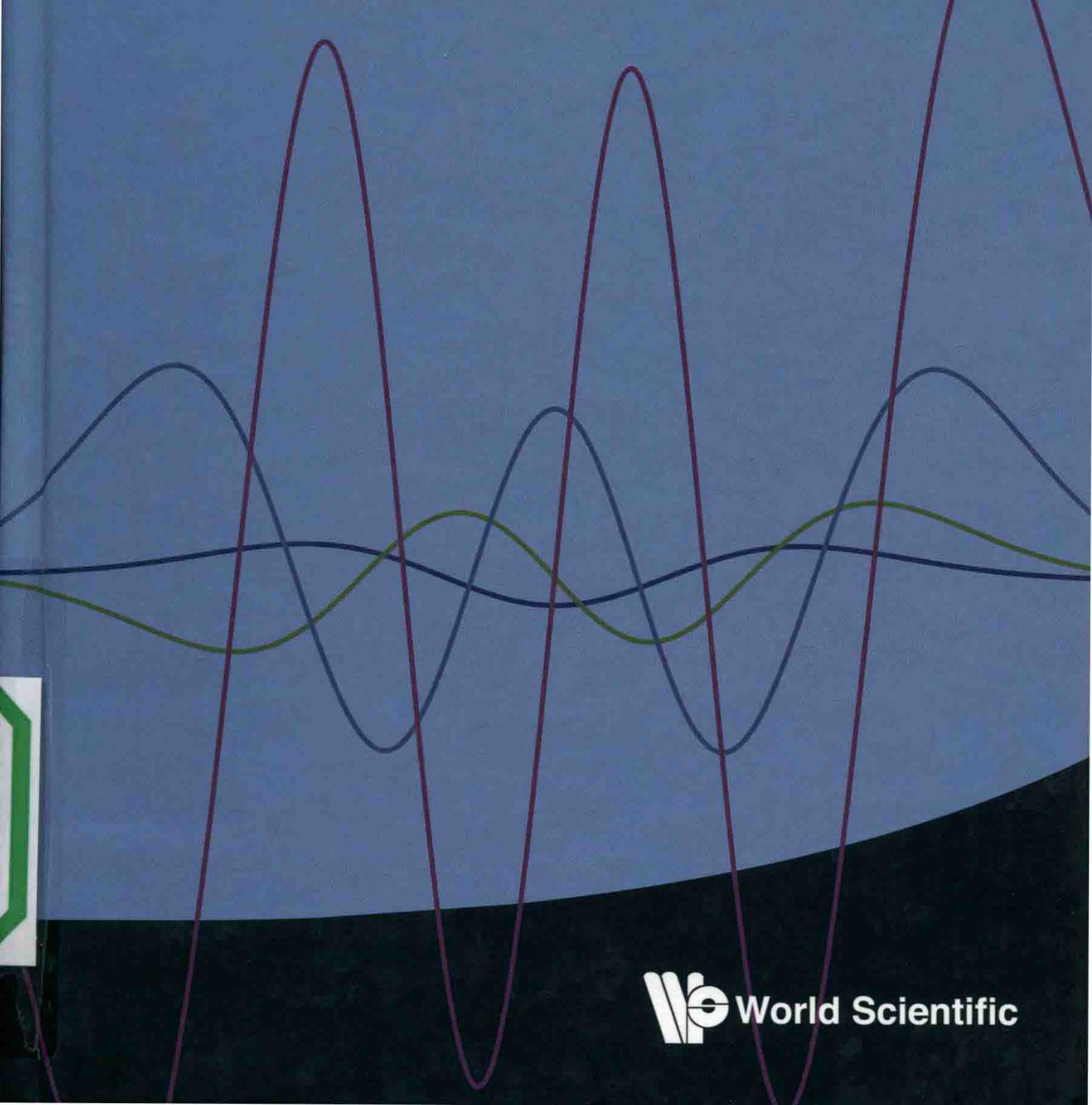
The Classical Orthogonal Polynomials

Brian George Spencer Doman



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This book defines sets of orthogonal polynomials and derives a number of properties satisfied by any such set. It continues by describing the classical orthogonal polynomials and the additional properties they have.

The first chapter defines the orthogonality condition for two functions. It then gives an iterative process to produce a set of polynomials which are orthogonal to one another and then describes a number of properties satisfied by any set of orthogonal polynomials. The classical orthogonal polynomials arise when the weight function in the orthogonality condition has a particular form. These polynomials have a further set of properties and in particular satisfy a second order differential equation.

Each subsequent chapter investigates the properties of a particular polynomial set starting from its differential equation.

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Preface

The Classical Orthogonal Polynomials have been studied extensively since the first set, the Legendre Polynomials, were discovered by Legendre in 1784. They frequently arise in the mathematical treatment of model problems in the Physical Sciences, often arising as solutions of ordinary differential equations subject to certain conditions imposed by the model. We shall not concern ourselves here with the physical applications. We shall be concentrating solely on their mathematical properties. This monograph derives a number of their basic properties together with some less well-known results.

The first chapter provides a survey of some general properties satisfied by any set of orthogonal polynomials. It starts by defining the inner product of two functions $f(x)$ and $g(x)$ as the integral of the product of these two functions multiplied by a non-negative weight function $w(x)$ over an interval (a, b), where a and b can be both finite, or one or both of infinite size. If this integral is zero, the functions $f(x)$ and $g(x)$ are said to be orthogonal. The functions we shall be considering are polynomials of arbitrary order.

The chapter continues by showing how for any given weight function we can use the orthogonality condition to produce a unique polynomial set by an iterative process and gives an example of this process. It shows that the orthogonality condition leads to a number of properties satisfied by any set of orthogonal polynomials. One of these is that members of any such set of polynomials satisfies a three-term recurrence relation. It also indicates that any set of polynomials satisfying such a recurrence relation forms an orthogonal set.

The first chapter then describes the particular choices of weight functions and domains which define the three classes of classical orthogonal polynomials. A number of additional properties of these classical orthogonal polynomials are then deduced. In particular it is shown that each of the polynomials satisfies a second order differential equation.

Each subsequent chapter focusses on a particular orthogonal polynomial set starting from the viewpoint of its differential equation. It shows that solutions of this differential equation which satisfy certain conditions are polynomials and that these polynomials form an orthogonal set. It then describes in detail a number of the properties outlined in chapter 1 together with further interesting properties.

A number of the polynomials have the Gamma Function $\Gamma(z)$ as part of their definition. The Gamma Function is defined in the General Appendix and the properties used in this monograph derived. The Beta Function $B(a, b)$ and the Hypergeometric Function ${}_2F_1(a, b; c; z)$ are also defined and the properties which are used in the earlier chapters described.

This monograph is an expanded version of a series of projects devised for undergraduate mathematicians at Liverpool University.

Contents

Chapter 1

Definitions and General Properties

1.1 Introduction

The classical orthogonal polynomials arise in a number of practical situations and models, often as solutions to differential equations arising from boundary value problems. We shall be collecting together and examining a number of their properties without detailed reference to their applications.

In this chapter we look at the definition for a set of orthogonal polynomials and describe a process for their generation and a number of their main properties. We then specialise to the classical orthogonal polynomials and deduce an additional number of general properties and show in particular that they all satisfy a second order differential equation. It is this approach from a differential equation which often arises in practical applications.

In later chapters we examine the properties of each of the individual polynomial sets from a different viewpoint, starting from the differential equation.

1.2 Definition of Orthogonality

The scalar or inner product of two functions $f(x)$ and $g(x)$ is defined by the integral

$$\int_a^b w(x)f(x)g(x)dx, \tag{1.2.1}$$

where $w(x) \geq 0$, for $a \leq x \leq b$.

This is a generalisation of the idea of a scalar product of two finite dimensional vectors to an infinite dimensional "function space". If this scalar product is zero, we say that the functions $f(x)$ and $g(x)$ are orthogonal. Here we shall be looking at functions $R_n(x)$ which are polynomials of order n.

If these n th order polynomials $R_n(x)$ satisfy the orthogonality relation

$$\int_a^b w(x)R_n(x)R_m(x)dx = 0 \qquad \text{for} \qquad n \neq m, \tag{1.2.2}$$

where $w(x)$ is a weight function which is non-negative in the interval (a, b) and is such that the integral is well-defined for all finite order polynomials $R_n(x)$, these polynomials form a set of orthogonal polynomials. It is clear that

$$\int_a^b w(x)[R_n(x)]^2 dx = h_n \geq 0 \tag{1.2.3}$$

since the integrand is everywhere ≥ 0 for $a < x < b$.

1.3 Gram-Schmidt Orthogonalisation Procedure

For a given weight function $w(x)$, this is an inductive procedure to generate a set of orthogonal polynomials starting from the zeroth order polynomial $R_0(x) = 1$. The procedure works by using the orthogonality condition to determine the coefficients of the powers of x in the polynomial $R_{n+1}(x)$ using all of the previously determined $R_m(x), 0 \leq m \leq n$.

The procedure consists of the following steps:
Set $R_0(x) = 1$.
Set $R_1(x) = x + a_{1,0}$. The constant $a_{1,0}$ is determined by the orthogonality condition:

$$\int_a^b w(x)\big(x + a_{1,0}\big)dx = 0 = \int_a^b x\, w(x)dx + a_{1,0}\int_a^b w(x)dx. \tag{1.3.1}$$

If the integral $\int_a^b x\, w(x)dx = 0$, then $a_{1,0} = 0$.
Set $R_2(x) = x^2 + a_{2,1}x + a_{2,0}$. The constants $a_{2,0}$ and $a_{2,1}$ are determined by the conditions that $R_2(x)$ is orthogonal to $R_1(x)$ and $R_0(x)$ that is:

$$\int_a^b w(x)\big(x^2 + a_{2,1}x + a_{2,0}\big)dx = 0 \tag{1.3.2}$$

and

$$\int_a^b w(x)\big(a_{1,0} + x\big)\big(x^2 + a_{2,1}x + a_{2,0}\big)dx = 0, \tag{1.3.3}$$