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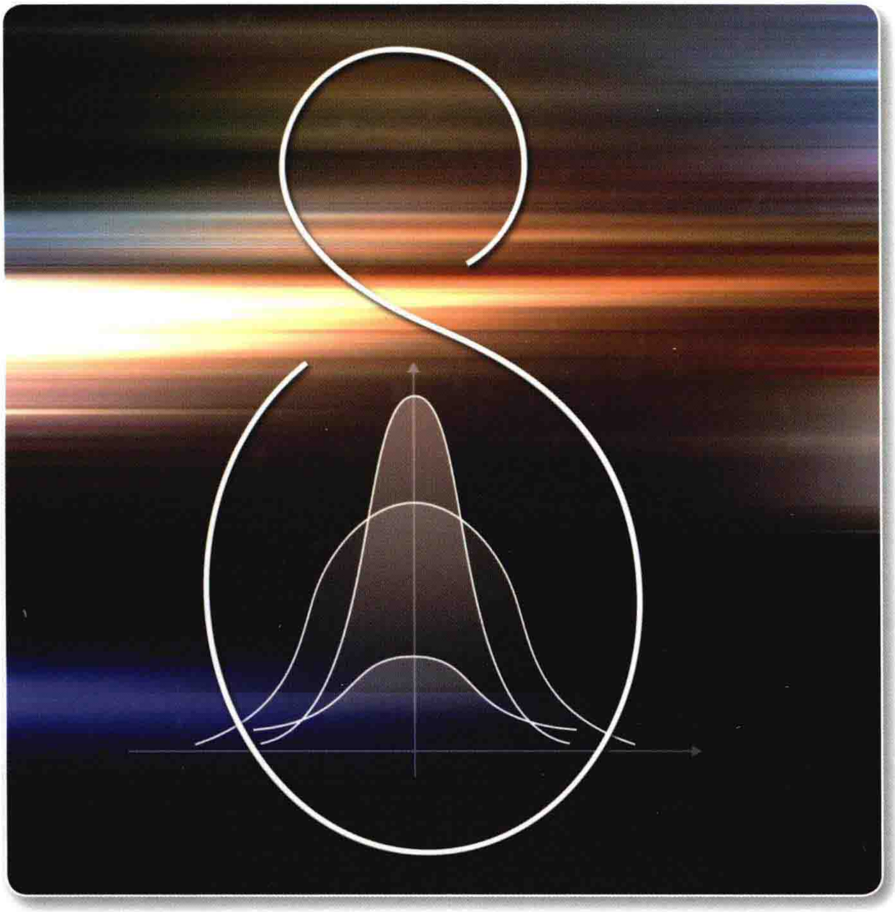
WILEY-VCH

Distribution Theory

With Applications in Engineering and Physics



Petre P. Teodorescu, Wilhelm W. Kecs, and Antonela Toma

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Preface

The solution to many theoretical and practical problems is closely connected to the methods applied, and to the mathematical tools which are used. In the mathematical description of mechanical and physical phenomena, and in the solution of the corresponding boundary value and limit problems, difficulties may appear owing to additional conditions. Sometimes, these conditions result from the limited range of applicability of the mathematical tool which is involved; in general, such conditions may be neither necessary nor connected to the mechanical or physical phenomenon considered.

The methods of classical mathematical analysis are usually employed, but their applicability is often limited. Thus, the fact that not all continuous functions have derivatives is a severe restriction imposed on the mathematical tool; it affects the unity and the generality of the results. For example, it may lead to the conclusion of the nonexistence of the velocity of a particle at any moment during the motion, a conclusion which obviously is not true.

On the other hand, the development of mechanics, of theoretical physics and particularly of modern quantum mechanics, the study of various phenomena of electromagnetism, optics, wave propagation and the solution of certain boundary value problems have all brought about the introduction of new concepts and computations, which cannot be justified within the frame of classical mathematical analysis.

In this way, in 1926 Dirac introduced the delta function (denoted by δ), which from a physical point of view, represents the density of a load equal to unity located at one point. A formalism has been worked out for the function, and its use justifies and simplifies various results. Except for a small number of incipient investigations, it was only during the 1960s that the theory of distributions was included as a new chapter of functional analysis. This theory represents a mathematical tool applicable to a large class of problems, which cannot be solved with the aid of classical analysis. The theory of distributions thus eliminates the restrictions which are not imposed by the physical phenomenon and justifies procedure and results, e.g., those corresponding to the continuous and discontinuous phenomena, which can thus be stated in a unitary and general form.

This monograph presents elements of the theory of distributions, as well as theorems with possibility of application. While respecting the mathematical rigor, a large number of applications of the theory of distributions to problems of general Newtonian mechanics, as well as to problems pertaining to the mechanics of deformable solids, are presented in a systematic manner; special stress is laid upon the introduction of corresponding mathematical models.

Some notions and theorems of Newtonian mechanics are stated in a generalized form; the effect of discontinuities on the motion of a particle and its mechanical interpretation is thus emphasized.

Particular stress is laid upon the mathematical representation of concentrated and distributed loads; in this way, the solution of the problems encountered in the mechanics of deformable solids may be obtained in a unitary form.

Newton's fundamental equation, the equations of equilibrium and of motion of the theory of elasticity are presented in a modified form, which includes the boundary and the initial conditions. In this case, the Fourier and the Laplace transforms may be easily applied to obtain the fundamental solutions of the corresponding differential equations; the use of the convolution product allows the expression of the boundary-value problem solutions for an arbitrary load.

Concerning the mechanics of deformable solids, not only have classical elastic bodies been taken into consideration, but also viscoelastic ones, that is, stress is put into dynamical problems: vibrations and propagation of waves.

Applications in physics have been described (acoustics, optics and electrostatics), as well as in electrotechnics.

The aim of the book is to draw attention to the possibility of applying modern mathematical methods to the study of mechanical and physical phenomena and to be useful to mathematicians, physicists, engineers and researchers, which use mathematical methods in their field of interest.

P.P. Teodorescu, W.W. Kecs, A. Toma *Bucharest, 22 August 2012*

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Introduction to the Distribution Theory

1.1
Short History

The theory of distributions, or of generalized functions, constitutes a chapter of functional analysis that arose from the need to substantiate, in terms of mathematical concepts, formulae and rules of calculation used in physics, quantum mechanics and operational calculus that could not be justified by classical analysis. Thus, for example, in 1926 the English physicist P.A.M. Dirac [1] introduced in quantum mechanics the symbol $\delta(x)$, called the Dirac delta function, by the formulae

$$\delta(x) = \begin{cases} 0\,, & x \neq 0 \\ \infty\,, & x = 0 \end{cases}\,, \quad \int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1\,. \tag{1.1}$$

By this symbol, Dirac mathematically described a material point of mass density equal to the unit, placed at the origin of the coordinate axis.

We notice immediately that $\delta(x)$, called the impulse function, is a function not in the sense of mathematical analysis, as being zero everywhere except the origin, but that its integral is null and not equal to unity.

Also, the relations $x\delta(x) = 0, \mathrm{d}H(x)/\mathrm{d}x = \delta(x)$ do not make sense in classical mathematical analysis, where

$$H(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0 \end{cases}$$

is the Heaviside function, introduced in 1898 by the English engineer Oliver Heaviside.

The created formalism regarding the use of the function δ and others, although it was in contradiction with the concepts of mathematical analysis, allowed for the study of discontinuous phenomena and led to correct results from a physical point of view.

All these elements constituted the source of the theory of distributions or of the generalized functions, a theory designed to justify the formalism of calculation used in various fields of physics, mechanics and related techniques.

Distribution Theory, First Edition. Petre P. Teodorescu, Wilhelm W. Kecs, and Antonela Toma

In 1936, S.L. Sobolev introduced distributions (generalized functions) in an explicit form, in connection with the study of the Cauchy problem for partial differential equations of hyperbolic type.

The next major event took place in 1950–1951, when L. Schwartz published a treatise in two volumes entitled "Théory des distributions" [2]. This book provided a unified and systematic presentation of the theory of distributions, including all previous approaches, thus justifying mathematically the calculation formalisms used in physics, mechanics and other fields.

Schwartz's monograph, which was based on linear functionals and on the theory of locally convex topological vector spaces, motivated further development of many chapters of mathematics: the theory of differential equations, operational calculus (Fourier and Laplace transforms), the theory of Fourier series and others.

Properties in the sense of distributions, such as the existence of the derivative of any order of a distribution and in particular of the continuous functions, the convergence of Fourier series and the possibility of term by term derivation of the convergent series of distributions, led to important technical applications of the theory of distributions, thus removing some restrictions of classical analysis.

The distribution theory had a significant further development as a result of the works developed by J. Mikusiński and R. Sikorski [3], M.I. Guelfand and G.E. Chilov [4, 5], L. Hörmander [6, 7], A. H. Zemanian [8], and so on.

Unlike the linear and continuous functionals method used by Schwartz to define distributions, J. Mikusiński and R. Sikorski introduced the concept of distribution by means of fundamental sequences of continuous functions.

This method corresponds to the spirit of classical analysis and thus it appears clearly that the concept of distribution is a generalization of the notion of function, which justifies the term generalized function, mainly used by the Russian school.

Other mathematicians, such as H. König, J. Korevaar, Sebastiano e Silva, and I. Halperin have defined the notion of distribution by various means (axiomatic, derivatives method, and so on).

Today the notion of distribution is generalized to the concept of a hyperfunction, introduced by M. Sato, [9, 10], in 1958. The hyperdistributions theory contains as special cases the extensions of the notion of distribution approached by C. Roumieu, H. Komatsu, J.F. Colombeu and others.

1.2 Fundamental Concepts and Formulae

For the purpose of distribution theory and its applications in various fields, we consider some function spaces endowed with a convergence structure, called fundamental spaces or spaces of test functions.

1.2.1
Normed Vector Spaces: Metric Spaces

We denote by Γ either the body $\mathbb{R}$ of real numbers or the body $\mathbb{C}$ of complex numbers and by $\mathbb{R}_+$, $\mathbb{R}^+$, $\mathbb{N}_0$ the sets $\mathbb{R}_+ = [0,\infty), \mathbb{R}^+ = (0,\infty), \mathbb{N}_0 = \{0,1,2,\ldots,n,\ldots\}$.

Let E, F be sets of abstract objects. We denote by $E \times F$ the direct product (Cartesian) of those two sets; where the symbol "$\times$" represents the direct or Cartesian product.

Definition 1.1 The set E is called a vector space with respect to Γ, and is denoted by (E, Γ), if the following two operations are defined: the sum, a mapping $(x, y) \to x + y$ from $E \times F$ into E, and the product with scalars from Γ, the mapping $(\lambda, x) \to \lambda x$ from $\Gamma \times E$ into E, having the following properties:

1. $\forall x, y \in E\,, \quad x + y = y + x\,;$
2. $\forall x, y, z \in E\,, \quad (x + y) + z = x + (y + z)\,;$
3. $\exists 0 \in E\,, \quad \forall x \in E\,, \quad x + 0 = x\,,$ (0 is the null element) ;
4. $\forall x \in E\,, \quad \exists x' = -x \in E\,, \quad x + (-x) = 0\,;$
5. $\forall x \in E\,, \quad 1 \cdot x = x\,;$
6. $\forall \lambda, \mu \in \Gamma\,, \quad \forall x \in E\,, \quad \lambda(\mu x) = (\lambda\mu)x\,;$
7. $\forall \lambda, \mu \in \Gamma\,, \quad \forall x \in E\,, \quad (\lambda + \mu)x = \lambda x + \mu x\,;$
8. $\forall \lambda \in \Gamma\,, \quad \forall x, y \in E\,, \quad \lambda(x + y) = \lambda x + \lambda y\,.$

The vector space (E, Γ) is real if $\Gamma = \mathbb{R}$ and it is complex if $\Gamma = \mathbb{C}$. The elements of (E, Γ) are called points or vectors.

Let X be an upper bounded set of real numbers, hence there is $M \in \mathbb{R}$ such that for all $x \in X$ we have $x \le M$. Then there exists a unique number $M^* = \sup X$, which is called the lowest upper bound of X, such that

1. $\forall x \in X\,, \quad x \le M^*\,;$
2. $\forall a \in \mathbb{R}\,, \quad a < M^*\,, \quad \exists x \in X$ such that $x \in (a, M^*]\,.$

Similarly, if Y is a lower bounded set of real numbers, that is, if there is $m \in \mathbb{R}$ such that for all $x \in Y$ we have $x \ge m$, then there exists a unique number $m^* = \inf X$, which is called the greatest lower bound of Y, such that

1. $\forall x \in Y\,, \quad x \ge m^*\,;$
2. $\forall b \in \mathbb{R}\,, \quad b > m^*\,, \quad \exists x \in Y$ such that $x \in [m^*, b)\,.$

Example 1.1 The vector spaces $\mathbb{R}^n$, $\mathbb{C}^n$, $n \ge 2$ Let us consider the n-dimensional space $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times). Two elements $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, are said to be equal, $x = y$, if $x_i = y_i$, $i = \overline{1, n}$.

Denote $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$, $\alpha \in \mathbb{R}$, then $\mathbb{R}^n$ is a real vector space, also called n-dimensional real arithmetic space.

The n-dimensional complex space $\mathbb{C}^n$ may be defined in a similar manner. The elements of this space are ordered systems of n complex numbers. The sum and product operations performed on complex numbers are defined similarly with those in $\mathbb{R}^n$.

Definition 1.2 Let (X, Γ) be a real or complex vector space. A norm on (X, Γ) is a function $\|\cdot\| : X \to [0, \infty)$ satisfying the following three axioms:

1. $\forall x \in X\,, \quad \|x\| > 0$ for $x \neq 0$, $\|0\| = 0\,;$
2. $\forall \lambda \in \Gamma\,, \quad \forall x \in X\,, \quad \|\lambda x\| = |\lambda| \|x\|\,;$
3. $\forall x, y \in X\,, \quad \|x + y\| \leq \|x\| + \|y\|\,.$

The vector space (X, Γ) endowed with the norm $\|\cdot\|$ will be called a normed vector space and will be denoted as $(X, \Gamma, \|\cdot\|)$.

The following properties result from the definition of the norm:

$$\|x\| \geq 0\,, \quad \forall x \in X\,,$$
$$|\|x_1\| - \|x_2\|| \leq \|x_1 - x_2\|\,, \quad \forall x_1, x_2 \in X\,,$$
$$\forall \alpha_i \in \Gamma\,, \quad \forall x_i \in X\,, \quad \|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \leq |\alpha_1| \|x_1\| + \cdots + |\alpha_n| \|x_n\|\,.$$

Definition 1.3 Let (X, Γ) be a vector space. We call an inner product on (X, Γ) a mapping $\langle \cdot, \cdot \rangle : E \to \Gamma$ that satisfies the following properties:

1. Conjugate symmetry: $\forall x \in X$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
2. Homogeneity: $\forall \alpha \in \Gamma, \forall x, y \in E, \langle \alpha x, y \rangle = \alpha \langle y, x \rangle$;
3. Additivity: $\forall x, y, z \in X, \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$;
4. Positive-definiteness: $\forall x \in X, \langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

An inner product space $(X, \langle \cdot, \cdot \rangle)$ is a space containing a vector space (X, Γ) and an inner product $\langle \cdot, \cdot \rangle$.

Conjugate symmetry and linearity in the first variable gives

$$\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a} \overline{\langle y, x \rangle} = \overline{a} \langle x, y \rangle\,,$$
$$\langle x, y + z \rangle = \overline{\langle y + z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle\,,$$

so an inner product is a sesquilinear form. Conjugate symmetry is also called Hermitian symmetry.

In the case of $\Gamma = \mathbb{R}$, conjugate-symmetric reduces to symmetric, and sesquilinear reduces to bilinear. Thus, an inner product on a real vector space is a positive-definite symmetric bilinear form.

Proposition 1.1 In any inner product space $(X, \langle \cdot, \cdot \rangle)$ the Cauchy–Schwarz inequality holds:

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}\,, \quad \forall x, y \in X\,, \tag{1.2}$$

with equality if and only if x and y are linearly dependent.