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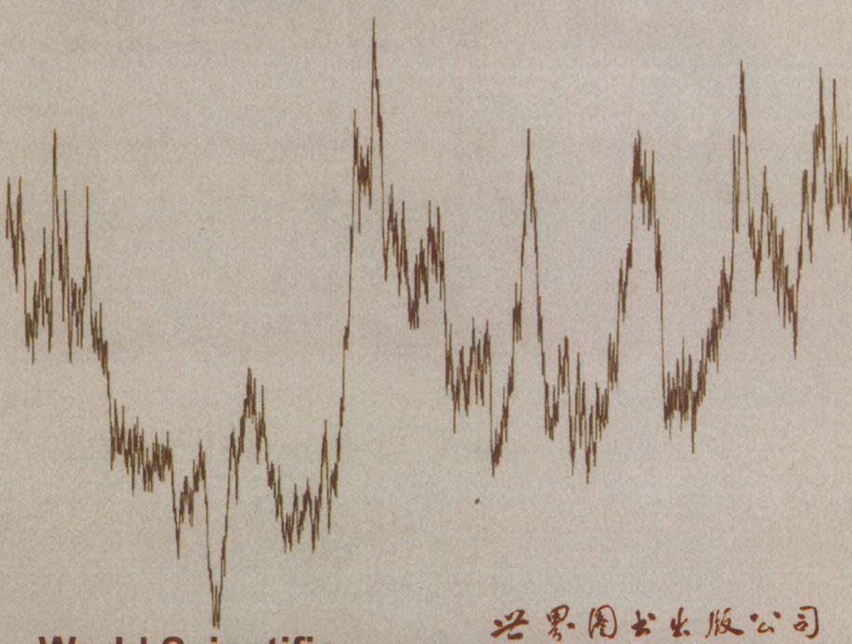
ELEMENTARY STOCHASTIC CALCULUS

with Finance in View

随机分析基础

Thomas Mikosch



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ELEMENTARY STOCHASTIC CALCULUS

with Finance in View

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Preface

Ten years ago I would not have dared to write a book like this: a non-rigorous treatment of a mathematical theory. I admit that I would have been ashamed, and I am afraid that most of my colleagues in mathematics still think like this. However, my experience with students and practitioners convinced me that there is a strong demand for popular mathematics.

I started writing this book as lecture notes in 1992 when I prepared a course on stochastic calculus for the students of the Commerce Faculty at Victoria University Wellington (New Zealand). Since I had failed in giving tutorials on portfolio theory and investment analysis, I was expected to teach something I knew better. At that time, staff members of economics and mathematics departments already discussed the use of the Black and Scholes option pricing formula; courses on stochastic finance were offered at leading institutions such as ETH Zürich, Columbia and Stanford; and there was a general agreement that not only students and staff members of economics and mathematics departments, but also practitioners in financial institutions should know more about this new topic.

Soon I realized that there was not very much literature which could be used for teaching stochastic calculus at a rather elementary level. I am fully aware of the fact that a combination of "elementary" and "stochastic calculus" is a contradiction in itself. Stochastic calculus requires advanced mathematical techniques; this theory cannot be fully understood if one does not know about the basics of measure theory, functional analysis and the theory of stochastic processes. However, I strongly believe that an interested person who knows about elementary probability theory and who can handle the rules of integration and differentiation is able to understand the main ideas of stochastic calculus. This is supported by my experience which I gained in courses for economics, statistics and mathematics students at VUW Wellington and the Department of Mathematics in Groningen. I got the same impression as a lecturer of crash courses on stochastic calculus at the Summer School of the

Swiss Association of Actuaries in Lausanne 1994, the Workshop on Financial Mathematics in Groningen 1997 and at the University of Leuven in May 1998.

Various colleagues, friends and students had read my lecture notes and suggested that I extend them to a small book. Among those are Claudia Klüppelberg and Paul Embrechts, my coauthors from a book about extremal events, and David Vere-Jones, my former colleague at the Institute of Statistics and Operations Research in Wellington. Claudia also proposed to get in contact with Ole Barndorff-Nielsen who is the editor of the probability series of World Scientific. I am indebted to him for encouraging me throughout the long process of writing this book.

Many colleagues and students helped in proofreading parts of the book at various stages. In particular, I would like to thank Leigh Roberts from Wellington, Bojan Basrak and Diemer Salome from Groningen. Their criticism was very helpful. I am most grateful to Carole Proctor from Sussex University. She was a constant source of inspiration, both on stylistic and mathematical issues. I also take pleasure in thanking the Department of Mathematics at the University of Groningen, my colleagues and students for their much appreciated support.

Thomas Mikosch Groningen, June 1, 1998

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Reader Guidelines

This book grew out of lecture notes for a course on stochastic calculus for economics students. When I prepared the first lectures I realized that there was no adequate textbook treatment for non-mathematicians. On the other hand, there was and indeed is an increasing demand to learn about stochastic calculus, in particular in economics, insurance, finance, econometrics. The main reason for this interest originates from the fact that this mathematical theory is the basis for pricing financial derivatives such as options and futures. The fundamental idea of Black, Scholes and Merton from 1973 to use Itô stochastic calculus for pricing and hedging of derivative instruments has conquered the real world of finance; the *Black–Scholes formula* has been known to many people in mathematics and economics long before Merton and Scholes were awarded the Nobel prize for economics in 1997.

For whom is this book written?

In contrast to the increasing popularity of financial mathematics, its theoretical basis is by no means trivial. Who ever tried to read the first few pages of a book on stochastic calculus will certainly agree. Tools from measure theory and functional analysis are usually required.

In this book I have tried to keep the mathematical level low. The reader will not be burdened with measure theory, but it cannot be avoided altogether. Then we will have to rely on heuristic arguments, stressing the underlying ideas rather than technical details. Notions such as measurable function and measurable set are not introduced, and therefore the formulation and proof of various statements and results are necessarily incomplete or non-rigorous. This may sometimes discourage the mathematically oriented reader, but for those, excellent mathematical textbooks on stochastic calculus exist.

In discussions with economists and practitioners from banks and insurance companies I frequently listened to the argument: "Itô calculus can be understood *only* by mathematicians." It is the main objective of this book to overcome this superstition.

Every mathematical theory has its roots in real life. Therefore the notions of Itô integral, Itô lemma and stochastic differential equation can be explained to anybody who ever attended courses on elementary calculus and probability theory: physicists, chemists, biologists, actuaries, engineers, economists, ... In the course of this book the reader will learn about the basic rules of stochastic calculus. Finally, you will be able to solve some simple stochastic differential equations, to simulate these solutions on a computer and to understand the mathematical ideology behind the modern theory of option pricing.

What are the prerequisites for this book?

You should be familiar with the rules of integration and differentiation. Ideally, you also know about differential equations, but it is not essential. You must know about elementary probability theory. Chapter 1 will help you to recall some facts about probability, expectation, distribution, etc., but this will not be a proper basis for the rest of the book. You would be advised to read one of the recommended books on probability theory, if this is new to you, before you attempt to read this book.

How should you read this book?

It depends on your knowledge of probability theory. I recommend that you browse through the "boxes" of Chapter 1. If you know everything that is written there, you can start with Chapter 2 on Itô stochastic calculus and continue with Chapter 3 on stochastic differential equations.

You cannot proceed in this way if you are not familiar with the following basic notions: stochastic process, Brownian motion, conditional expectation and martingale. There is no doubt that you will struggle with the notion of conditional expectation, unless you have some background on measure theory. Conditional expectation is one of the key notions underlying stochastic integration.

The ideal reader can handle simulations on a computer. Computer graphs of Brownian motion and solutions to stochastic differential equations will help you to experience the theory. The theoretical tools for these simulations will be provided in Sections 1.3.3 and 3.4.

I have not included lists of exercises, but I will ask you various questions in the course of this book. Try to answer them. They are not difficult, but they aim at testing the level of your understanding.

Besides Sections 1.3–1.5, the core material is contained in Chapter 2 and the first sections of Chapter 3. Chapter 2 provides the construction of the Itô integral and a heuristic derivation of the Itô lemma, the chain rule of stochastic calculus. In Chapter 3 you will learn how to solve some simple stochastic differential equations. Section 3.3 on linear stochastic differential equation is mainly included in order to exercise the use of the Itô lemma. Section 3.4 will be interesting to those who want to visualize solutions to stochastic differential equations.

Chapter 4 is for those readers who want to see how stochastic calculus enters financial applications. Prior knowledge of economic theory is not required, but we will introduce a minimum of economic terminology which can be understood by everybody. If you can read through Section 4.1 on option pricing without major difficulties as regards stochastic calculus, you will have passed the examination on this course on elementary stochastic calculus.

At the end of this book you may want to know more about stochastic calculus and its applications. References to more advanced literature are given in the Notes and Comments at the end of each section. These references are not exhaustive; they do not include the theoretically most advanced textbook treatments, but they can be useful for the continuation of your studies.

You are now ready to start. Good luck!

T.M.

1

Preliminaries

In this chapter we collect some basic facts needed for defining stochastic integrals. At a first reading, most parts of this chapter can be skipped, provided you have some basic knowledge of probability theory and stochastic processes. You may then want to start with Chapter 2 on Itô stochastic calculus and recall some facts from this chapter if necessary.

In Section 1.1 we recall elementary notions from probability theory such as *random variable, random vector, distribution, distribution function, density, expectation, moment, variance* and *covariance*. This small review cannot replace a whole course on probability, and so you are well recommended to consult your old lecture notes or a standard textbook. Section 1.2 is about *stochastic processes*. A stochastic process is a natural model for describing the evolution of real-life processes, objects and systems in time and space. One particular stochastic process plays a central rôle in this book: *Brownian motion*. We introduce it in Section 1.3 and discuss some of its elementary properties, in particular the non-differentiability and the unbounded variation of its sample paths. These properties indicate that Brownian sample paths are very irregular, and therefore a new, stochastic calculus has to be introduced for integrals with respect to Brownian motion.

In Section 1.4 we shortly review *conditional expectations*. Their precise definition is based on a deep mathematical theory, and therefore we only give some intuition on this concept. The same remark applies to Section 1.5, where we introduce an important class of stochastic processes: the *martingales*. It includes Brownian motion and indefinite Itô integrals as particular examples.

1.1 Basic Concepts from Probability Theory

1.1.1 Random Variables

The outcome of an experiment or game is random. A simple example is coin tossing: the possible outcomes "head" or "tail" are not predictable in the sense that they appear according to a random mechanism which is determined by the physical properties of the coin. A more complicated experiment is the stock market. There the random outcomes of the brokers' activities (which actually represent economic tendencies, political interests and their own instincts) are for example share prices and exchange rates. Another game is called "competition" and can be watched where products are on sale: the price of 1 kg bananas, say, is the outcome of a game between the shop owners, on the one hand, and between the shop owners and the customers, on the other hand.

The scientific treatment of an experiment requires that we assign a number to each random outcome. When tossing a coin, we can write "1" for "head" and "0" for "tail". Thus we get a *random variable* $X = X(\omega) \in \{0, 1\}$, where ω belongs to the *outcome space* $\Omega = \{\text{head,tail}\}$. The value of a share price of stock is already a random number, and so is the banana price in a greengrocers. These numbers $X(\omega)$ provide us with information about the experiment, even if we do not know who plays the game or what drives it.

Mathematicians make a clear cut between reality and a mathematical model: they define an abstract space Ω collecting all possible outcomes ω of the underlying experiment. It is an abstract space, i.e. it does not really matter what the ωs are. In mathematical language, the *random variable* $X = X(\omega)$ is nothing but a real-valued function defined on Ω.

The next step in the process of abstraction from reality is the probabilistic description of the random variable X:

Which are the most likely values $X(\omega)$, what are they concentrated around, what is their spread?

To approach these problems, one first collects "good" subsets of Ω, the *events*, in a class $\mathcal{F}$, say. In advanced textbooks $\mathcal{F}$ is called a *σ-field* or *σ-algebra*; see p. 62 for a precise definition. Such a class is supposed to contain all interesting events. What could $\mathcal{F}$ be for coin tossing? Certainly, $\{\omega : X(\omega) = 0\} = \{\text{tail}\}$ and $\{\omega : X(\omega) = 1\} = \{\text{head}\}$ must belong to $\mathcal{F}$, but also the union, difference, intersection of any events in $\mathcal{F}$, the set $\Omega = \{\text{head,tail}\}$ and its complement, the empty set $\emptyset$. This is a trivial example, but it shows what $\mathcal{F}$ should be like: if $A \in \mathcal{F}$, so is its complement A^c, and if $A, B \in \mathcal{F}$, so are $A \cap B$, $A \cup B$, $A \cup B^c$, $B \cup A^c$, $A \cap B^c$, $B \cap A^c$, etc.