ANNALS
OF
DISCRETE
MATHEMATICS
3

advances in graph theory

B. Bollobás, editor

NORTH-HOLLAND

ANNALS OF DISCRETE MATHEMATICS 3

ADVANCES IN GRAPH THEORY

Edited by

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1978

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM • NEW YORK • OXFORD

Reprinted from the journal *Annals of Discrete Mathematics*, Volume 3

North-Holland ISBN for this Volume 0 7204 0843 1

Published by:

NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM • NEW YORK • OXFORD

Sole distributors for the U.S.A. and Canada:

ELSEVIER NORTH-HOLLAND, INC.
52 VANDERBILT AVENUE
NEW YORK, NY 10017

PRINTED IN THE NETHERLANDS

ADVANCES IN GRAPH THEORY

annals of discrete mathematics

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM • NEW YORK • OXFORD

FOREWORD

The Cambridge Combinatorial Conference was held at Trinity College from 12 to 14 May 1977, under the auspices of the Department of Pure Mathematics and Mathematical Statistics. Twenty two of the participants, many from abroad, were invited to give talks. This volume consists of most of the papers they presented, together with two additional articles which are closely connected with the themes of the conference. The opportunity was taken, where necessary, to revise and amend the papers, each of which has been thoroughly refereed. It is a pleasure to acknowledge the rapid and efficient work of both referees and authors.

This volume is dedicated to Professor W.T. Tutte in acknowledgement of his great contributions to graph theory and combinatorics. Professor Tutte had spent two months in Cambridge, with the financial support of the Science Research Council, and the date of the conference was arranged to coincide with his sixtieth birthday. On Friday 13 May a celebration dinner was held in Trinity College. Professor P.W. Duff, Regius Professor of Civil Law Emeritus, who was Professor Tutte's tutor while he was a student at Trinity, proposed a most memorable toast which received an equally memorable reply.

Several of the papers were quickly and efficiently retyped by Mrs. J.E. Scutt. The editorial burden was greatly relieved by the excellent work of Mr. A.G. Thomason.

Béla Bollobás
Cambridge

3 August, 1977

W.T. TUTTE

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LINEAR SEPARATION OF DOMINATING SETS IN GRAPHS*

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The class of finite undirected graphs G having the property that there exist real positive numbers associated to their vertices so that a set of vertices is dominating if and only if the sum of the corresponding weights exceeds a certain threshold θ is characterized: (a) by forbidden induced subgraphs; (b) by the linearity of a certain partial order on the vertices of G; (c) by the global structure of G. The class properly includes that of threshold graphs and is properly included in that of perfect graphs.

1. Introduction, notations, main results

We shall consider in this paper only finite, simple, loopless, undirected graphs $G = (V, E)$ (where V is the vertex set of G, and E is the edge set of G). The terminology follows that in [1] or [5].

For any $x \in V$, we shall denote by $N(x)$ the set of vertices adjacent to x and by $M(x)$ the set of vertices of G not belonging to $x \cup N(x)$ (for simplicity we shall usually put x instead of $\{x\}$).

The edgeless graph on k vertices will be denoted by I_k. The complete graph with k vertices will be denoted by K_k. The complement of the perfect matching of $2k$ vertices will be denoted by J_{2k}. (Note that $I_0 = K_0 = J_0 = \emptyset$, $I_1 = K_1$, $I_2 = J_2$.)

Following Zykov's terminology [8], for two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, with $V_1 \cap V_2 = \emptyset$, we shall define their direct sum $G_1 + G_2$ as being $(V_1 \cup V_2,\ E_1 \cup E_2)$ and their direct product $G_1 \times G_2$ as being $(V_1 \cup V_2,\ E_1 \cup E_2 \cup E_{12})$, where E_{12} is the set of all edges linking points in V_1 to points in V_2.

A subset S of the vertex set V of a graph G is called a *dominating* set of G (in abbreviation S dom G) if any vertex $x \notin S$ is adjacent to at least one vertex $y \in S$. A vertex v is called *universal* (or *dominating*) if $\{v\}$ dom G. Every set containing a dominating set is dominating.

A subset S of V is called an *independent* set of G when the induced subgraph G_S is edgeless. Every subset of an independent set is independent.

*This research has been carried out at the University of Waterloo (December 1976) and completed at the University of Grenoble (March 1977).

A maximal independent set of G is a minimal dominating set of G. The converse is generally not true. A *domistable* graph is a graph such that every minimal dominating set is independent.

A *domishold* graph is a graph having the property that there exist positive real numbers associated to their vertices so that S is dominating if and only if the sum of the corresponding "weights" of vertices of S exceeds a certain threshold θ.

Examples and counterexamples. Both I_n and K_n are domishold and domistable graphs. Each weight is 1, and the thresholds θ are n (for I_n) and 1 (for K_n). For $p>1$, the graph J_{2p} is domishold (each weight is 1, and the threshold θ is 2), but not domistable.

Let $H_1 = K_2 + K_2$, let H_2 be the simple path on 4 vertices, and let $H_3 = I_3 \times I_3$, $H_4 = (I_1 + K_2) \times I_3$, $H_5 = (I_1 + K_2) \times (I_1 + K_2)$ (see Fig. 1). It is easy to notice that none of the graphs in Fig. 1 are domishold.

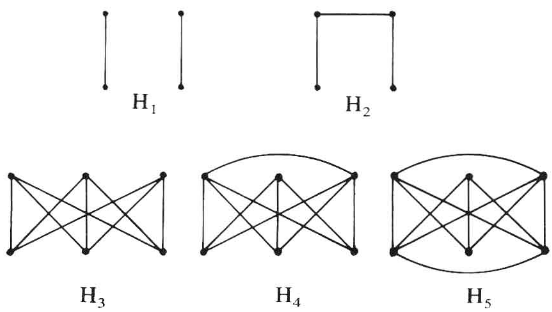

Fig. 1.

Let us define now a binary relation δ_G on the vertex set V of G, by putting $x\delta_G y$ $(x, y \in V)$ iff

$$(S \text{ dom } G,\ x \notin S,\ y \in S) \Rightarrow ((S \setminus y) \cup x) \text{ dom } G.$$

We shall say that "x is at least as dominating as y", or that "x can replace y".

Lemma 1.1. *δ_G is a reflexive and transitive relation (i.e. a preorder).*

Proof. The reflexivity is obvious. Assume $i\delta_G j$ and $j\delta_G k$ (i, j, k – distinct), and let S be a dominating set of G, containing k, but not i. If $j \notin S$, then $((S \setminus k) \cup j)$ dom G (because $j\delta_G k$) and does not contain i; therefore $((S \setminus k) \cup i)$ dom G because i can replace j. If $j \in S$ then $((S \setminus j) \cup i)$ dom G, contains k but not j. So $((((S \setminus j) \cup i) \setminus k) \cup j)$ dom G, i.e. $((S \setminus k) \cup i)$ dom G. In both cases $i\delta_G k$.

The main results of this paper are the following:

Theorem 1.2. *The following properties are equivalent:*

(i) *G is domishold.*
(ii) *The preorder δ_G is linear.*
(iii) *G has no induced subgraph isomorphic to H_1, H_2, H_3, H_4 or H_5.*

(iv) *G is built from the empty graph by the repeated application of* $G' \to G''$ *where*

$$G'' = (G' + I_p) \times K_q \times J_{2r} \qquad (p+q+r \neq 0).$$

Corollaries. (a) *Every induced subgraph of a domishold graph is domishold* (because of (iii)).

(b) *Every domishold graph is perfect* (follows from [6] where it is proved that a graph without any induced subgraphs isomorphic to H_2 is perfect).

Theorem 1.3. *G is a domishold graph iff, the vertex set V of G can be partitioned into three (possibly empty) subsets* V_1, V_2, V_3 ($|V_3|$ *being even*) *inducing respectively the graphs* $I_{|V_1|}$, $K_{|V_2|}$, $J_{|V_3|}$ *with the following properties*:

Any vertex of V_2 *is adjacent to any vertex of* V_3.

For any $i \in V_1$, $N(i) \cap V_3$ *induces the complement of a perfect matching* J_{2k} *with* $2k = |N(i) \cap V_3|$.

The elements of V_1 *can be indexed so that*

$$N(i_1) \supseteq N(i_2) \supseteq \cdots \supseteq N(i_{|V_1|}).$$

The proofs of these results are given in Section 2.

Section 3 deals with connections between threshold and domishold graphs. Consider an arbitrary threshold graph G, and let L be an arbitrary subset of vertices, inducing a complete subgraph in G. A one to one correspondence is established between the set of all pairs (G, L) (taken for all threshold graphs G and all their complete subsets L) and the set of all domishold graphs.

Section 4 deals with Boolean aspects of the previously obtained results and with algorithms for recognizing domishold graphs.

2. Proof of the main results

Proposition 2.1. *If G is domishold, then* δ_G *is a linear preorder.*

Proof. Indeed, if G is domishold and a_i are the weights associated to its vertices, then it is obvious that for any pair of vertices j, k one of the relations $j\delta_G k$ (if $a_j \geq a_k$) or $k\delta_G j$ (if $a_k \geq a_j$) must hold.

A vertex m of G is called *maximal* if it is maximal with respect to $\delta_G(m\delta_G i, \forall i \in V_G)$.

Remarks. (1) Any dominating vertex is maximal.

(2) If a graph has a dominating vertex, then every maximal vertex is dominating.

Lemma 2.2. *Let G be a graph such that the corresponding preorder* δ_G *is linear and let m be a maximal vertex of it.*

If m is neither an isolated nor a dominating vertex then every pair $\{x, y\}$ with $x \in N(m)$, $y \in M(m)$ is a dominating set of G and every vertex $y \in M(m)$ is dominating in $G_{V \setminus m}$.

Proof. Let S be a maximal independent set of G not containing m (its existence is guaranteed by the fact that m is not isolated). Thus S dom G. Let i be any element of S. Since $m \notin S$ and $m \delta_G i$ we must have $(S \setminus i) \cup m$ dom G; i is not adjacent to any vertex of $S \setminus i$, hence it must be adjacent to m. So $S \subseteq N(m)$.

Now, if $\{x, y\}$ is such that $x \in N(m)$, $y \in M(m)$ then x is adjacent to y (otherwise $\{x, y\}$ is included in a maximal independent set of G not containing m and not included in $N(m)$). This means that $\{x, y\}$ dom G (every $x' \in N(m)$ is adjacent to y, every $y' \in M(m)$ is adjacent to x and m is adjacent to x).

But m can replace x and $\{m, y\}$ dom G. Hence every y' in $M(m)$ is adjacent to y, proving the Lemma.

Lemma 2.3. *If δ_G is linear, m is a maximal vertex of G and G_m the subgraph induced by $V \setminus m$, then the preorder δ_{G_m} is also linear.*

Proof. Let $i, j \in V \setminus m$ and assume $i \delta_G j$. Let S be a dominating set of G_m containing j but not i. Assume first that m is isolated in G. Then $S \cup m$ dom G contains j but not i. Hence $((S \cup m) \setminus j) \cup i$ dom G and by deleting m, $(S \setminus j) \cup i$ dom G_m showing that $i \delta_{G_m} j$. If m is a dominating vertex of G then S dom G and $(S \setminus j) \cup i$ dom G_m, showing that $i \delta_{G_m} j$. Finally let us consider the case where m is neither isolated nor dominating. If i or j belongs to $M(m)$ then $i \delta_{G_m} j$ (or $j \delta_{G_m} i$) because by Lemma 2.2 i(resp. j) is dominating and so maximal in G_m. If i and j belong to $N(m)$ then S dom G and $(S \setminus j) \cup i$ dom G does not contain m so that $(S \setminus j) \cup i$ dom G_m. Hence $i \delta_{G_m} j$.

Lemma 2.4. *If δ_G is linear, m is a nonisolated vertex of it, and $i, j \in V \setminus m$ such that $i \in M(m) \cap M(j)$, then m is not a maximal vertex of G.*

Proof. Otherwise (by Lemma 2.2) $i \in M(m)$ must be dominating in $G_{V \setminus m}$, which is impossible since i is not adjacent to j ($j \neq m$).

Lemma 2.5. *If δ_G is linear, $m \in V$, i and j are adjacent vertices in $M(m)$ and if $h, k, l \in N(m)$ are such that $l \in M(h) \cap M(k)$, then m is not a maximal vertex of G.*

Proof. Assume m is maximal. Since it is neither isolated nor dominating, it follows from Lemma 2.2 that $\{h, i\}$ and $\{k, j\}$ are dominating sets of G. However $\{i, j\}$ and $\{k, h\}$ are not dominating (because $m \in M(i) \cap M(j)$ and $l \in M(h) \cap M(k)$). Hence neither $j \delta_G h$ nor $h \delta_G j$ hold, in contradiction with the assumed linearity of δ_G.

Proposition 2.6. *A graph G having the property that the preorder δ_G is linear, cannot have any induced subgraph isomorphic to H_1, H_2, H_3, H_4 or H_5.*

Proof. Assume that G with linear preorder δ_G has an induced subgraph H isomorphic to an H_t $(t = 1, \ldots, 5)$.

By removing a maximal vertex $m \notin H$ (if possible) and continuing this process as many times as possible, we shall eventually arrive (by Lemma 2.3) to a graph G' (with linear preorder) having a maximal vertex m in its induced subgraph H.

If $t = 1, 2$ or 3 then we can find two vertices n $(\neq m)$ and p such that $p \in M(m) \cap M(n)$. By Lemma 2.4, m is not maximal (a contradiction). If $t = 4$ or 5 then $H = (I_1 + K_2) \times H'$ (where $H' = I_3 (t = 4)$ or $H' = I_1 + K_2 (t = 5)$).

By the same argument as above $m \notin K_2$. Similarly, if $H' = I_3$, $m \notin I_3$. So we may suppose $m = I_1$. Then $K_2 \subseteq M(m)$ while H' is a subset of $N(m)$. It follows now, from Lemma 2.5, that m is not maximal. In any case, we have a contradiction.

Lemma 2.7. (Wolk [7].) *If G is a connected graph without a dominating vertex, then the complementary graph $\bar{G}$ contains an induced subgraph isomorphic to H_1 or H_2.*

Lemma 2.8. *If G has no isolated or dominating vertex and no induced subgraph isomorphic to H_t $(t = 1, 2, \ldots, 5)$ then its complement $\bar{G}$ has an isolated edge (i.e. an edge which is not adjacent to any other edge).*

Proof.[1] $\bar{G}$ has no dominating vertex. If $\bar{G}$ is connected then by Lemma 2.7, G contains a subgraph isomorphic to H_1 or H_2 (a contradiction). If $\bar{G}$ is not connected then every connected component has at least two vertices (G has no dominating vertex). If one component has exactly two vertices the lemma is proved. Otherwise each component contains a subgraph isomorphic to one of the following

Hence, G contains a subgraph isomorphic to $L_1 \times L_2$ where the L_i $(i = 1, 2)$ are I_3 or $I_1 + K_2$. Thus G contains a subgraph isomorphic to H_t $(t = 3$ or 4 or $5)$.

Lemma 2.9. *If G is not empty and has no induced subgraph isomorphic to H_t $(t = 1, 2, \ldots, 5)$ then G has one of the forms*

$$G = G' + I_1,$$

$$G = G' \times K_1,$$

$$G = G' \times J_2,$$

where G' has no induced subgraph isomorphic to any H_t $(t = 1, \ldots, 5)$.

[1] The use of Wolk's result in the present proof was recommended by Ch.Payan and has produced a substantial simplification over our original proof.

Proof. The decomposition follows from Lemma 2.8; the fact that G' has no induced subgraph isomorphic to H_t is obvious.

Proposition 2.10. *If G has no induced subgraph isomorphic to H_t $(t=1,2,\ldots,5)$ then G is built from the empty graph by the repeated application of $G' \to G''$ where*

$$G'' = (G' + I_p) \times K_q \times J_{2l} \quad \textit{with} \quad p+q+l \neq 0.$$

Proof. Obvious from the repeated application of Lemma 2.9, from the associativity and commutativity of $+$ and $\times$, and from the following relations:

$$I_p = I_1 + I_1 + I_1 + \cdots + I_1 \quad (p \text{ times}),$$
$$K_q = K_1 \times K_1 \times K_1 \times \cdots \times K_1 \quad (q \text{ times}),$$
$$J_{2l} = J_2 \times J_2 \times J_2 \times \cdots \times J_2 \quad (l \text{ times}).$$

Proposition 2.11. *If G is built from the empty graph by the repeated application of $G' \to G''$ defined above then G is domishold.*

Proof. The empty graph is obviously domishold. Assume now that $G = (G' + I_p) \times K_q \times J_{2r}$, and that G' is domishold. Let w_l represent the weight of the vertices $l \in V_{G'}$, and w_0 the threshold for G'. Let $w^* = \frac{1}{2}\min_l w_l$. We can always assume that $2w^* \leqslant w_0$, since otherwise $G' = K_q$, and we could take all weights w_l $(l \in V_{G'})$, as well as w_0, equal to 1 (in which case again $2w^* \leqslant w_0$). Let us also put $W = 1 + \sum_{l \in V_{G'}} w_l$ and let us define $\hat{w}_0 = w_0 + pW$ and

$$\hat{w}_i = \begin{cases} w_i & i \in V_G', \\ W & i \in I_p, \\ w_0 + pW & i \in K_q, \\ w_0 + pW - w^* & i \in J_{2r}. \end{cases}$$

The "weights" $\hat{w}_i$ and the "threshold" $\hat{w}_0$ of G characterize the dominating sets of G. Indeed, any minimal dominating set D of G is of one of the following three types: (i) $D = \{k\}$, $k \in K_q$; (ii) $D = \{j, e\}$, with $j \in J_{2r}$, $j \neq e$, and $e \in J_{2r} \cup I_p \cup V_{G'}$; (iii) $D = D' \cup I_p$, where D' is a minimal dominating set of G'.

Proof of Theorem 1.2. Follows from Propositions 2.1, 2.6, 2.10, 2.11.

Proof of Theorem 1.3. *Necessity.* From property (iv) of Theorem 1.2, we can define $G_0, G_1, \ldots, G_t$ with

$$G_0 = \emptyset \qquad G_t = G$$

and

$$G_{i+1} = (G_i + I_{p_i}) \times K_{q_i} \times J_{2r_i} \qquad (i = 0, 1, \ldots, t-1).$$

Putting

$$V_1 = \bigcup_i I_{p_i} \qquad V_2 = \bigcup_i K_{q_i} \qquad V_3 = \bigcup_i J_{2r_i},$$

it is clear by induction that this partition has the desired properties.
Sufficiency. By induction. If G has the prescribed properties it is obvious that if $V_1 \neq \emptyset$ then $i_{|V_1|}$ is either isolated (and after its elimination we get a graph G' with the same properties), or is adjacent to a vertex k in V_2 (k is dominating and its elimination leads to a graph G' with the same properties), or is adjacent to a non-adjacent pair $\{j, j'\}$ of V_3 so that $G = G' \times J_2$ with G' having the same properties.

If $V_1 = \emptyset$ then every $k \in V_2$ (in case of $V_2 \neq \emptyset$) is dominating and G_{V-k} has the same properties.

If $V_2 = \emptyset$ then $G = J_{2r}$ is domishold.

3. Threshold and domishold graphs

We shall recall that, as in [2], by a *threshold graph* we shall mean a graph such that real non negative numbers can be associated to its vertices so that two vertices are adjacent iff the sum of their weights exceeds a certain threshold. Alternatively, a graph is threshold iff there exist real numbers associated to its vertices so that the sum of these numbers associated to vertices belonging to an independent set (a dependent set) is $<$ (is $\geqslant$) than a certain threshold. Several characterizations of such graphs can be found in [2].

We recall also that, as in [3], by a *split graph* we shall mean a graph whose vertex set V can be partitioned in two (possibly empty) subsets V_1, V_2 such that V_1 induces $I_{|V_1|}$ and V_2 induces $K_{|V_2|}$.

Theorem 3.1. *Every threshold graph is domishold and has all the following properties*:

(a) *It has no induced square* $[I_2 \times I_2]$.
(b) *It is split.*
(c) *It is domistable.*
(d) *It is an interval graph.*

Conversely a domishold graph having any one of the mentioned properties is threshold.

Proof. It has been proved in [2] that a threshold graph is characterized by the absence of induced subgraphs isomorphic to H_1, H_2 and $I_2 \times I_2$. From this, it follows that it has no subgraph isomorphic to H_t $(t = 1, 2, \ldots, 5)$. Hence it is domishold and satisfies (a).

In [2] it is proved that a threshold graph is split (b). Moreover if i and j are two adjacent vertices of a threshold graph, we have:

$$N(i) \subseteq N(j) \cup j \quad \text{or} \quad N(j) \subseteq N(i) \cup i.$$

Therefore a minimal dominating set will never contain both i and j, and hence it is independent, proving (c).

Finally it has been proved [2] that in a threshold graph (having split structure $V_1 \cup V_2$) one can index the elements of V_1 in such a way that $N(i_1) \subseteq N(i_2) \subseteq \cdots \subseteq N(i_{|V_1|})$. Associate to each element $i_\alpha \in V_1$ the interval $[\alpha, \alpha]$ and to each element $k \in V_2$ the interval $[m_k, |V_1|+1]$ where m_k is the least integer such that $k \in N(i_{m_k})$ (if any), and otherwise $m_k = |V_1|+1$. It is easy to see that the corresponding interval graph is isomorphic to the original one, proving (d).

Conversely a domishold graph without an induced square $I_2 \times I_2$ (and of course without H_1, H_2) is threshold. A split graph has no square and if it is domishold, it is threshold.

If a graph G is domishold and domistable then a maximal vertex m (for δ_G) is either dominating or isolated. Otherwise by Lemma 2.2 any set $\{x, y\}$ with $x \in N(m)$, $y \in M(m)$ is minimal dominating (the minimality follows from the fact that y is obviously not dominating, and neither is x, otherwise m should be dominating). Hence by removing m, we get again a domistable and domishold graph. By induction it follows that the original graph is threshold.

Finally if an interval graph is domishold then it does not contain a square. Indeed assume there exists a square and $[a, b]$, $[c, d]$ are the corresponding intervals associated to two opposite vertices of this square. We have $[a, b] \cap [c, d] = \emptyset$. Obviously, the two intervals $[e, f]$, $[g, h]$ associated to the other two vertices of the square must intersect both $[a, b]$ and $[c, d]$ and therefore intersect each other (a contradiction).

Definition. Let i be a vertex of a graph G. The *i-duplication of G* is the graph G' obtained by adding a new vertex i' to V_G with $N(i') = N(i)$. Conversely, we shall say that G is the *(i, i')-fusion* of G'.

We can extend this definition to *W-duplication* of G ($W \subseteq V_G$) by duplicating sequentially each vertex of W (this operation does not depend on the order of duplications).

Also if $U \subseteq V_G$, induces the complement of a perfect matching ($J_{|U|}$) and if every pair (i, i') of non adjacent vertices in U have the same neighbourhood ($N(i) = N(i')$), by the *U-fusion* of G, we mean the graph obtained by the sequential repetition of all the (i, i')-fusions of G.

Theorem 3.2. *If G is a domsihold graph and S a maximal subset of V_G inducing a subgraph $J_{|S|}$ then the S-fusion of G is threshold.*

Conversely if G is threshold and L a subset of a maximal subset of V_G inducing a clique of G then the L-duplication of G is a domishold graph.

Proof. A direct consequence of Theorem 1.3.

Remark. Despite the fact that the class of domishold graphs includes properly that one of threshold graphs, this theorem seems to point to a (to us) surprising similarity between threshold and domishold graphs.

4. Boolean aspects and algorithms

4.1. *Recognizing domishold graphs*

It is easy now to construct a procedure for the recognition of domishold graphs; the time needed by this procedure will be polynomial in the number n of vertices. The procedure can start by searching for isolated vertices and eliminating them. When no more isolated vertices can be found, the procedure could search for dominating vertices. After repeating the above two steps as many times as possible we shall obtain a graph without isolated or dominating vertices; in this graph we shall look for two non-adjacent vertices, both of which are linked to every other vertex. The graph is domishold if and only if the above three steps can be repeated until the total exhaustion of the vertex set.

4.2. *Recognizing the linear separator of a domishold graph*

A linear inequality

$$\sum_{i=1}^{n} w_i x_i \geqslant w_0, \qquad x_i \in \{0, 1\} \qquad (i = 1, \ldots, n)$$

is called *domigraphic* if there exists a domishold graph of n vertices such that the w_i's are the weights of the vertices, and w_0 is the threshold. In other words, the inequality holds if and only if $(x_1, \ldots, x_n)$ is the characteristic vector of a dominating set. We can obviously assume that $w_1 \geqslant \cdots \geqslant w_n$.

The condition

$$\sum_{i=1}^{n} w_i \geqslant w_0$$

is obviously necessary for a linear inequality to be domigraphic. In the case $n = 1$, it is sufficient too. For $n = 2$, this condition along with $(w_2 < w_0) \Rightarrow (w_1 < w_0)$ are again sufficient.

Theorem 4.1. *The inequality $\sum_{i=1}^{n} w_i x_i \geqslant w_0$ $(n \geqslant 3)$ is domigraphic if and only if one of the following conditions hold:*

(i) $w_1 \geqslant w_0$ *and* $\sum_{i=2}^{n} w_i x_i \geqslant w_0$ *is domigraphic;*
(ii) $w_1 < w_0$, $\sum_{i=2}^{n} w_i < w_0$ *and* $\sum_{i=2}^{n} w_i x_i \geqslant w_0 - w_1$ *is domigraphic;*
(iii) $w_1 < w_0$, $w_2 + w_n \geqslant w_0$ *and* $\sum_{i=3}^{n} w_i x_i \geqslant w_0$ *is domigraphic.*