The Selected Works of Roderick S. C. Wong

Volume 3

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Contents

Volume 3

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Asymptotic Expansion of the Krawtchouk Polynomials and their Zeros

Wei-Yuan Qiu and Roderick Wong

(Communicated by Lawrence Zalcman)

Abstract. Let $K_n^N(x;p,q)$ be the Krawtchouk polynomials and $\mu = N/n$. An asymptotic expansion is derived for $K_n^N(x;p,q)$, when x is a fixed number. This expansion holds uniformly for μ in $[1,\infty)$, and is given in terms of the confluent hypergeometric functions. Asymptotic approximations are also obtained for the zeros of $K_n^N(x;p,q)$ in various cases depending on the values of p,q and μ.

Keywords. Krawtchouk polynomials, asymptotic expansions, confluent hypergeometric functions, zeros.

2000 MSC. 33C45, 41A60.

1. Introduction

Let $p>0$, $q>0$ and $p+q=1$, and let N be a positive integer. By the binomial expansion, we have

$$(1.1)\qquad (1-pw)^{N-x}(1+qw)^x = \sum_{n=0}^{\infty} K_n^N(x;p,q)w^n,$$

where

$$(1.2)\qquad K_n^N(x;p,q) = \sum_{k=0}^{n} \binom{N-x}{n-k}\binom{x}{k}(-p)^{n-k}q^k.$$

It is clear that $K_n^N(x) \equiv K_n^N(x;p,q)$ is a polynomial in x of degree n. The polynomials $\{K_n^N(x)\}_{n=0}^N$ are known as the Krawtchouk polynomials, and they

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form an orthogonal system on the discrete set $\{0, 1, 2, \ldots, N\}$ with the weights

$$\rho(x) = \binom{N}{x} p^x q^{N-x}, \qquad x = 0, 1, \ldots, N. \tag{1.3}$$

It is well known that the Krawtchouk polynomials are related to the Hamming association scheme in coding theory (see, e.g. [5, 13, 15, 21]), and Lloyd's Theorem [12, 15] states that the existence of a perfect code in the Hamming metric corresponds to the Krawtchouk polynomials having integer zeros. Recently, there has been considerable interest in the asymptotics of Krawtchouk polynomials, when the degree n grows to infinity. For instance, in [20], Sharapudinov obtained the asymptotic formula

$$\begin{aligned} &(2Npq\pi n!)^{1/2}(Npq)^{-n/2}\rho(\hat{x})e^{x^2/2}K_n^N(\hat{x}) \\ &\qquad = e^{-x^2/2}(2^n n!)^{-1/2}H_n(x) + \mathcal{O}(n^{7/4}N^{-1/2}), \end{aligned} \tag{1.4}$$

where $\hat{x} = Np + (2Npq)^{1/2}x$, $n = \mathcal{O}(N^{1/3})$, $x = \mathcal{O}(n^{1/2})$, and $H_n(x)$ is the Hermite polynomial. Furthermore, if the zeros of $K_n^N(\hat{x})$ are arranged in decreasing order: $\hat{x}_{1,N} > \hat{x}_{2,N} > \ldots > \hat{x}_{n,N}$, then he has shown that

$$\hat{x}_{n,N} = Np[1 - (2q/Np)^{1/2}x_1(n)] + \mathcal{O}(n^{7/4}) \tag{1.5}$$

uniformly with respect to $1 \leq n \leq \eta_N N^{1/4}$, $N = 1, 2, \ldots$, where $\{\eta_N\}$ is a sequence of positive numbers tending to zero as $N \to \infty$ and $x_1(n)$ is the smallest zero of the Hermite polynomial. Also, Dragnev and Saff [6, 7] have given the distribution of the zeros of Krawtchouk polynomials.

In [9], Ismail and Simeonov have investigated the asymptotic behavior of $K_n^N(nt)$ as $n \to \infty$, when $N/n = \gamma$ is a fixed constant independent of n. They divided the t-interval $0 < t < \gamma$ into several subintervals, and gave an asymptotic formula for $K_n^N(nt)$ in each of the subintervals by using the classical saddle point method [22]. Their formulae hold uniformly on compact subsets of each subinterval, but are not valid for $x = nt$ in any bounded interval in $[0, +\infty)$ since in this case t has to tend to zero as $n \to \infty$. When $\gamma \geq 1/p$, they have also proved that the zeros of $K_n^N(nt)$ are in the interval

$$\left(p\gamma - p + q - \sqrt{pq(\gamma-1)},\ p\gamma - p + q\sqrt{pq(\gamma-1)}\right). \tag{1.6}$$

More accurate bounds for the zeros have been given by Krasikov [10].

Recently, Li and Wong [14] have studied the uniform asymptotic behavior of $K_n^N(x)$ in the interval $0 < x < N$, as $n \to \infty$. With $\mu = n/N$ and $x = \lambda N$, they have derived an infinite asymptotic expansion for $K_n^N(\lambda N)$ as $n \to \infty$, which holds uniformly for μ and λ in any compact subinterval of $(0, 1)$. For fixed μ, their result covers the various asymptotic approximations given by Ismail and Simeonov [9]. Another treatment of uniform asymptotics of $K_n^N(x)$ with scaled variable $x = nt$ can be found in Baik, Kriecherbauer, McLaughlin and

Miller [3], where a steepest descent method for a Riemann-Hilbert problem is used. However, the results in both [9] and [3] are not valid for fixed or bounded x.

The purpose of this paper is to present an asymptotic expansion for $K_n^N(x)$ with fixed or bounded x, which is uniformly valid for n/N in $(0,1]$. This expansion is given in terms of the confluent hypergeometric functions. To state the result more precisely, we let $\mu = N/n$. When $n \to \infty$, we have the asymptotic expansion

$$\begin{aligned} K_n^N(x) \sim\ & (-1)^{n+1} p^{n-x} e^{-n\gamma} \\ & \cdot \left[\mathbf{N}(-n(\mu-1), x-n(\mu-1)+1, n\eta) \sum_{k=0}^{\infty} (-1)^k \frac{a_k}{(n\eta)^k} \right. \\ & \left. + \mathbf{N}'(-n(\mu-1), x-n(\mu-1)+1, n\eta) \sum_{k=0}^{\infty} (-1)^k \frac{b_k}{(n\eta)^k} \right], \end{aligned} \tag{1.7}$$

which holds uniformly for $1 \le \mu < \infty$ and for x in any bounded subinterval of $[0, \infty)$, where $\mathbf{N}$ denotes the confluent hypergeometric function given in [18, p. 255], γ and η are analytic functions of μ given in Section 3 below, and a_k and b_k can be calculated recursively; see (4.16)–(4.18).

Asymptotic approximations are also obtained for the zeros of Krawtchouk polynomials $K_n^{n\mu}(x; p, q)$ in various cases depending on the values of p, q and μ. Let $x_{n,k}$ denote the k-th zero of $K_n^{n\mu}(x)$, $k = 0, 1, 2, \ldots, n-1$, counted in increasing order

$$0 \le x_{n,0} < x_{n,1} < \ldots < x_{n,n-1} \le N.$$

As $n \to \infty$ we have

(i) if $n(\mu - 1/p)^2 \to \infty$ and $\mu > 1/p$ then each $x_{n,k}$ tends to infinity,
(ii) if $n(\mu - 1/p)^2 \to \infty$ and $\mu < 1/p$ then $x_{n,k}$ equals k up to an exponentially small error;
(iii) if $n(\mu - 1/p)^2$ is bounded then $x_{n,k}$ can be expressed in terms of the k-th zero of the parabolic cylinder function given in (4.29) below.

The precise statement of the result is given in Theorem 2 at the end of the paper.

2. The steepest descent method

By Cauchy's formula, we have from (1.1)

$$K_n^N(x) = \frac{1}{2\pi i} \int_C (1-pw)^{N-x} (1+qw)^x \frac{dw}{w^{n+1}}, \tag{2.1}$$

where C is a small, positively oriented, closed contour surrounding $w = 0$. Changing the variable in (2.1) to $t = 1/(1-pw)$ we have

$$K_n^N(x) = -\frac{p^{n-x}}{2\pi i} \int_{C'} \frac{(t-q)^x}{t^{N-n+1}(t-1)^{n+1}}\, dt, \tag{2.2}$$

where C' is a circle surrounding $t = 1$, oriented in the negative direction and with a radius less than 1.

If we set $\mu = N/n$, then $1 \le \mu < +\infty$ and (2.2) can be written in the form

$$K_n^{n\mu}(x) = (-1)^{n+1}\frac{p^{n-x}}{2\pi i}\int_{C'}\frac{(t-q)^x}{t(t-1)}e^{-nf(t,\mu)}dt, \tag{2.3}$$

where the phase function $f(t, \mu)$ is given by

$$f(t,\mu) = \log(1-t) + (\mu - 1)\log t. \tag{2.4}$$

When $\mu > 1$, the function $f(t, \mu)$ has two branch points $t = 0$ and $t = 1$. This function is analytic and single-valued in the t-plane with cuts along the intervals $(-\infty, 0]$ and $[1, \infty)$. For all logarithmic functions in (2.4), we choose the principal branch

$$\log\zeta = \log|\zeta| + i\arg\zeta, \qquad -\pi < \arg\zeta < \pi.$$

For later discussion, we also need to specify the values of the argument along the upper and lower edges of the cuts. If $t^{\pm} = u + i0^{\pm}$, $u < 0$, denote points on the upper and lower edges of the cut along $(-\infty, 0]$ respectively, then we choose

$$\arg t^{+} = \pi, \qquad \arg t^{-} = -\pi; \tag{2.5}$$

if $t^{\pm} = u + i0^{\pm}$, $u > 1$, are points on the upper and lower edges of the cut along $[1, \infty)$ respectively, we choose

$$\arg(1 - t^{+}) = -\pi, \qquad \arg(1 - t^{-}) = \pi. \tag{2.6}$$

When $\mu = 1$, the function $f(t, \mu)$ has only one branch point $t = 1$. By choosing the principal value of the logarithmic function indicated above, $f(t, \mu)$ is analytic in the cut plane $\mathbb{C} \setminus [1, \infty)$.

The amplitude function in the integral in (2.3) also has a branch point $t = q$ when x is not an integer. For definiteness of $(t - q)^x$, we choose a cut along $(-\infty, q]$ in the t-plane and take the branch of $(t - q)^x$ in the cut plane so that $(t - q)^x$ is real when $t > q$. Thus, if $t^{\pm} = u + i0^{\pm}$, $u < q$, are points along the upper and lower edges of the cut along $(-\infty, q]$, then

$$\arg(t^{\pm} - q)^x = \pm\pi x. \tag{2.7}$$

The derivative of $f(t, \mu)$ with respect to t is given by

$$f'(t,\mu) = \frac{\mu t - (\mu - 1)}{t(t-1)}. \tag{2.8}$$

For $1 < \mu < +\infty$, the function $f(t, \mu)$ has a saddle point at $t_0 = (\mu - 1)/\mu$. This saddle point lies in $0 < t_0 < 1$, and approaches 0, as $\mu \to 1$, which is one of the branch points of $f(t, \mu)$. Note that the saddle point will disappear when μ reaches 1; i.e. so the function $f(t, \mu)$ has no saddle point for $\mu = 1$.

At the saddle point t_0, we have

$$f(t_0, \mu) = (\mu - 1)\log(\mu - 1) - \mu\log\mu, \tag{2.9}$$

and

$$f''(t_0,\mu) = \frac{\mu^3}{1-\mu} < 0. \tag{2.10}$$

If x is not an integer, there is also a cut along $(-\infty, q]$ for the amplitude function. When $t_0 < q$, or equivalently $\mu < 1/p$, we let t_0^+ and t_0^- denote the point t_0 on the upper and lower edges of the cut respectively. By taking branches as in (2.5), (2.6) and (2.7), we have

$$f(t_0^+, \mu) = f(t_0^-, \mu)$$

but

$$(t_0^+ - q)^x = \overline{(t_0^- - q)^x} = (q - t_0)^x e^{i\pi x}.$$

The relevant path of steepest descent L for the integral (2.3) is given by

$$\operatorname{Im}(f(t,\mu) - f(t_0,\mu)) = \arg(1-t) + (\mu - 1)\arg t = 0 \tag{2.11}$$

and

$$\operatorname{Re}(f(t,\mu) - f(t_0,\mu)) \geq 0. \tag{2.12}$$

It is obvious that the path of steepest descent L is symmetric with respect to the real axis. Note that points in the interval $(0,1)$ on the real axis satisfy (2.11), but do not satisfy (2.12). Indeed, we have $\operatorname{Re}(f(t,\mu) - f(t_0,\mu)) \to -\infty$ as $t \to 0$ and $t \to 1$. Hence, the path of steepest descent L must be the path through t_0, perpendicular to the real axis at t_0 and going to infinity; see Figure 1. If we set $t = u + iv$, $v > 0$, then the half branch L_+ of L in the upper half plane is described by the equation

$$-\operatorname{arccot}\frac{1-u}{v} + (\mu - 1)\operatorname{arccot}\frac{u}{v} = 0. \tag{2.13}$$

The integration contour C' in (2.3) can be deformed into the oriented curve Γ shown in Figure 1. When $\mu \geq 1/p$, i.e. $t_0 \geq q$, Γ is exactly the path of steepest descent L. When $1 < \mu < 1/p$, i.e. $0 < t_0 < q$, Γ consists of the lower half branch L_- of L, the lower edge ℓ_- from t_0^- to q and the upper edge ℓ_+ from q to t_0^+ of the cut along $(-\infty, q]$, and the upper half branch L_+ of L; see Figure 1 (b). When $\mu = 1$, Γ consists of the lower and upper edges of the cut along $(-\infty, q]$, with two small semicircular indentations near 0, since 0 is a simple pole of the integrand in (2.3); see Figure 1 (c).

The integral (2.3) can now be written as

$$K_n^{n\mu}(x) = (-1)^{n+1} p^{n-x} I(n,\mu;x), \tag{2.14}$$

where

$$I(n,\mu;x) = \frac{1}{2\pi i}\int_\Gamma \frac{(t-q)^x}{t(t-1)} e^{-nf(t,\mu)}\,dt. \tag{2.15}$$

We need consider only the integral $I(n,\mu;x)$.

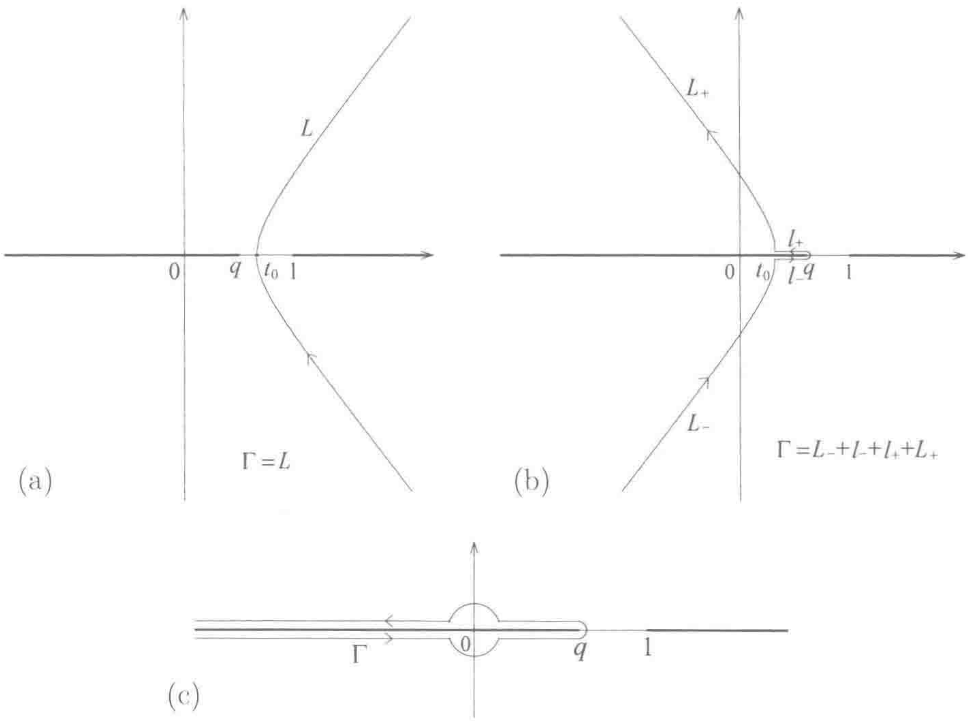

FIGURE 1. Path of steepest descent Γ: (a) $t_0 > q$; (b) $t_0 \le q$; (c) $t_0 = 0$.

3. Relation to confluent hypergeometric functions

To obtain the asymptotic expansion of the integral (2.15) for bounded $x \in [0, +\infty)$ which holds uniformly for $1 \le \mu < \infty$, we should compare (2.15) with a well-known special function. Note that the function $f(t, \mu)$ in (2.15) has a saddle point at t_0. This saddle point moves close to $t = q$, the branch point of the amplitude function, when μ gets close to $1/p$; it moves close to $t = 0$, the logarithmic singularity of $f(t, \mu)$, when μ gets close to 1. To find a function having the same properties as $f(t, \mu)$, we introduce the function

$$\begin{aligned} \mathbf{N}(a, c; z) &= \frac{1}{2\pi i} \int_{-\infty}^{(1+)} s^{a-1} (s-1)^{c-a-1} e^{zs}\, ds \\ &= \frac{1}{2\pi i} \int_{-\infty}^{(1+)} (s-1)^{c-a-1} e^{zs + (a-1)\log s}\, ds, \end{aligned} \tag{3.1}$$

where the integration path begins at $-\infty$, encircles the point $t = 1$ once in the positive direction, and then returns to $-\infty$; see Figure 2. The cut in the s-plane is taken along $(-\infty, 1]$, and we take the principal branches for the functions s^{a-1} and $(s-1)^{c-a-1}$ in the cut plane.