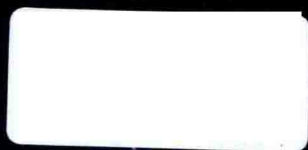


THE SELECTED WORKS OF RODERICK S. C. WONG



VOLUME 3

EDITORS

DAN DAI • HUI-HUI DAI
TONG YANG • DING-XUAN ZHOU

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RODERICK S. C. WONG

VOLUME 3

Contents

Preface		xi
Photos		xvii
Volume 1		
1.	The Asymptotic Behaviour of $\mu(z, \beta, \alpha)$, <i>Canad. J. Math.</i> , 21 (1969), 1013–1023.	1
2.	A Generalization of Watson's Lemma, <i>Canad. J. Math.</i> , 24 (1972), 185–208.	12
3.	Linear Equations in Infinite Matrices, <i>Linear Algebra and Appl.</i> , 7 (1973), 53–62.	36
4.	Asymptotic Solutions of Linear Volterra Integral Equations with Singular Kernels, <i>Trans. Amer. Math. Soc.</i> , 189 (1974), 185–200.	46
5.	On Infinite Systems of Linear Differential Equations, <i>Canad. J. Math.</i> , 27 (1975), 691–703.	62
6.	Error Bounds for Asymptotic Expansions of Hankel Transforms, <i>SIAM J. Math. Anal.</i> , 7 (1976), 799–808.	75
7.	Explicit Error Terms for Asymptotic Expansions of Stieltjes Transforms, <i>J. Inst. Math. Appl.</i> , 22 (1978), 129–145.	85
8.	Explicit Error Terms for Asymptotic Expansions of Mellin Convolutions, <i>J. Math. Anal. Appl.</i> , 72 (1979), 740–756.	102
9.	Asymptotic Expansion of Multiple Fourier Transforms, <i>SIAM J. Math. Anal.</i> , 10 (1979), 1095–1104.	119

-
10. Exact Remainders for Asymptotic Expansions of Fractional Integrals, *J. Inst. Math. Appl.*, **23**(1979), 139–147. 129
 11. Asymptotic Expansion of the Hilbert Transform, *SIAM J. Math. Anal.*, **11**(1980), 92–99. 138
 12. Error Bounds for Asymptotic Expansions of Integrals, *SIAM Rev.*, **22**(1980), 401–435. 146
 13. Distributional Derivation of an Asymptotic Expansion, *Proc. Amer. Math. Soc.*, **80**(1980), 266–270. 181
 14. On a Method of Asymptotic Evaluation of Multiple Integrals, *Math. Comp.*, **37**(1981), 509–521. 186
 15. Asymptotic Expansion of the Lebesgue Constants Associated with Polynomial Interpolation, *Math. Comp.*, **39**(1982), 195–200. 199
 16. Quadrature Formulas for Oscillatory Integral Transforms, *Numer. Math.*, **39**(1982), 351–360. 205
 17. Generalized Mellin Convolutions and Their Asymptotic Expansions, *Canad. J. Math.*, **36**(1984), 924–960. 215
 18. A Uniform Asymptotic Expansion of the Jacobi Polynomials with Error Bounds, *Canad. J. Math.*, **37**(1985), 979–1007. 252
 19. Asymptotic Expansion of a Multiple Integral, *SIAM J. Math. Anal.*, **18**(1987), 1630–1637. 281
 20. Asymptotic Expansion of a Double Integral with a Curve of Stationary Points, *IMA J. Appl. Math.*, **38**(1987), 49–59. 289
 21. Szegő's Conjecture on Lebesgue Constants for Legendre Series, *Pacific J. Math.*, **135**(1988), 157–188. 300
 22. Uniform Asymptotic Expansions of Laguerre Polynomials, *SIAM J. Math. Anal.*, **19**(1988), 1232–1248. 332
 23. Transformation to Canonical Form for Uniform Asymptotic Expansions, *J. Math. Anal. Appl.*, **149**(1990), 210–219. 349

-
24. Multidimensional Stationary Phase Approximation: Boundary Stationary Point, *J. Comput. Appl. Math.*, **30**(1990), 213–225. 359
 25. Two-Dimensional Stationary Phase Approximation: Stationary Point at a Corner, *SIAM J. Math. Anal.*, **22**(1991), 500–523. 372
 26. Asymptotic Expansions for Second-Order Linear Difference Equations, *J. Comput. Appl. Math.*, **41**(1992), 65–94. 396
 27. Asymptotic Expansions for Second-Order Linear Difference Equations, II, *Stud. Appl. Math.*, **87**(1992), 289–324. 426
 28. Asymptotic Behaviour of the Fundamental Solution to $\partial u / \partial t = -(-\Delta)^m u$, *Proc. Roy. Soc. London Ser. A*, **441**(1993), 423–432. 462
 29. A Bernstein-Type Inequality for the Jacobi Polynomial, *Proc. Amer. Math. Soc.*, **121**(1994), 703–709. 472
 30. Error Bounds for Asymptotic Expansions of Laplace Convolutions, *SIAM J. Math. Anal.*, **25**(1994), 1537–1553. 479

Volume 2

31. Asymptotic Behavior of the Pollaczek Polynomials and Their Zeros, *Stud. Appl. Math.*, **96**(1996), 307–338. 497
32. Justification of the Stationary Phase Approximation in Time-Domain Asymptotics, *Proc. Roy. Soc. London Ser. A*, **453**(1997), 1019–1031. 529
33. Asymptotic Expansions of the Generalized Bessel Polynomials, *J. Comput. Appl. Math.*, **85**(1997), 87–112. 542
34. Uniform Asymptotic Expansions for Meixner Polynomials, *Constr. Approx.*, **14**(1998), 113–150. 568
35. “Best Possible” Upper and Lower Bounds for the Zeros of the Bessel Function $J_\nu(x)$, *Trans. Amer. Math. Soc.*, **351**(1999), 2833–2859. 606

-
36. Justification of a Perturbation Approximation of the Klein–Gordon Equation, *Stud. Appl. Math.*, **102**(1999), 375–417. 633
37. Smoothing of Stokes’s Discontinuity for the Generalized Bessel Function. II, *Proc. Roy. Soc. London Ser. A*, **455**(1999), 3065–3084. 676
38. Uniform Asymptotic Expansions of a Double Integral: Coalescence of Two Stationary Points, *Proc. Roy. Soc. London Ser. A*, **456**(2000), 407–431. 696
39. Uniform Asymptotic Formula for Orthogonal Polynomials with Exponential Weight, *SIAM J. Math. Anal.*, **31**(2000), 992–1029. 721
40. On the Asymptotics of the Meixner–Pollaczek Polynomials and Their Zeros, *Constr. Approx.*, **17**(2001), 59–90. 759
41. Gevrey Asymptotics and Stieltjes Transforms of Algebraically Decaying Functions, *Proc. Roy. Soc. London Ser. A*, **458**(2002), 625–644. 791
42. Exponential Asymptotics of the Mittag–Leffler Function, *Constr. Approx.*, **18**(2002), 355–385. 811
43. On the Ackerberg–O’Malley Resonance, *Stud. Appl. Math.*, **110**(2003), 157–179. 842
44. Asymptotic Expansions for Second-Order Linear Difference Equations with a Turning Point, *Numer. Math.*, **94**(2003), 147–194. 865
45. On a Two-Point Boundary-Value Problem with Spurious Solutions, *Stud. Appl. Math.*, **111**(2003), 377–408. 913
46. Shooting Method for Nonlinear Singularly Perturbed Boundary-Value Problems, *Stud. Appl. Math.*, **112**(2004), 161–200. 945

Volume 3

47. Asymptotic Expansion of the Krawtchouk Polynomials and Their Zeros, *Comput. Methods Funct. Theory*, **4**(1)(2004), 189–226. 985
48. On a Uniform Treatment of Darboux's Method, *Constr. Approx.*, **21**(2005), 225–255. 1023
49. Linear Difference Equations with Transition Points, *Math. Comp.*, **74**(2005), 629–653. 1054
50. Uniform Asymptotics for Jacobi Polynomials with Varying Large Negative Parameters — A Riemann–Hilbert Approach, *Trans. Amer. Math. Soc.*, **358**(2006), 2663–2694. 1079
51. Uniform Asymptotics of the Stieltjes–Wigert Polynomials via the Riemann–Hilbert Approach, *J. Math. Pures Appl.*, **85**(5)(2006), 698–718. 1111
52. A Singularly Perturbed Boundary-Value Problem Arising in Phase Transitions, *European J. Appl. Math.*, **17**(6)(2006), 705–733. 1132
53. On the Number of Solutions to Carrier's Problem, *Stud. Appl. Math.*, **120**(3)(2008), 213–245. 1161
54. Asymptotic Expansions for Riemann–Hilbert Problems, *Anal. Appl. (Singap.)*, **6**(2008), 269–298. 1194
55. On the Connection Formulas of the Third Painlevé Transcendent, *Discrete Contin. Dyn. Syst.*, **23**(2009), 541–560. 1224
56. Hyperasymptotic Expansions of the Modified Bessel Function of the Third Kind of Purely Imaginary Order, *Asymptot. Anal.*, **63**(2009), 101–123. 1244
57. Global Asymptotics for Polynomials Orthogonal with Exponential Quartic Weight, *Asymptot. Anal.*, **64**(2009), 125–154. 1267

58.	The Riemann–Hilbert Approach to Global Asymptotics of Discrete Orthogonal Polynomials with Infinite Nodes, <i>Anal. Appl. (Singap.)</i> , 8 (2010), 247–286.	1297
59.	Global Asymptotics of the Meixner Polynomials, <i>Asymptot. Anal.</i> , 75 (2011), 211–231.	1337
60.	Asymptotics of Orthogonal Polynomials via Recurrence Relations, <i>Anal. Appl. (Singap.)</i> , 10 (2)(2012), 215–235.	1358
61.	Uniform Asymptotic Expansions for the Discrete Chebyshev Polynomials, <i>Stud. Appl. Math.</i> , 128 (2012), 337–384.	1379
62.	Global Asymptotics of the Hahn Polynomials, <i>Anal. Appl. (Singap.)</i> , 11 (3)(2013), 1350018.	1427
63.	Global Asymptotics of Stieltjes–Wigert Polynomials, <i>Anal. Appl. (Singap.)</i> , 11 (5)(2013), 1350028.	1474
	List of Publications by Roderick S. C. Wong	1487
	Permissions	1501

Asymptotic Expansion of the Krawtchouk Polynomials and their Zeros

Wei-Yuan Qiu and Roderick Wong

(Communicated by Lawrence Zalcman)

Abstract. Let $K_n^N(x; p, q)$ be the Krawtchouk polynomials and $\mu = N/n$. An asymptotic expansion is derived for $K_n^N(x; p, q)$, when x is a fixed number. This expansion holds uniformly for μ in $[1, \infty)$, and is given in terms of the confluent hypergeometric functions. Asymptotic approximations are also obtained for the zeros of $K_n^N(x; p, q)$ in various cases depending on the values of p, q and μ .

Keywords. Krawtchouk polynomials, asymptotic expansions, confluent hypergeometric functions, zeros.

2000 MSC. 33C45, 41A60.

1. Introduction

Let $p > 0$, $q > 0$ and $p + q = 1$, and let N be a positive integer. By the binomial expansion, we have

$$(1.1) \quad (1 - pw)^{N-x}(1 + qw)^x = \sum_{n=0}^{\infty} K_n^N(x; p, q)w^n,$$

where

$$(1.2) \quad K_n^N(x; p, q) = \sum_{k=0}^n \binom{N-x}{n-k} \binom{x}{k} (-p)^{n-k} q^k.$$

It is clear that $K_n^N(x) \equiv K_n^N(x; p, q)$ is a polynomial in x of degree n . The polynomials $\{K_n^N(x)\}_{n=0}^N$ are known as the Krawtchouk polynomials, and they

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form an orthogonal system on the discrete set $\{0, 1, 2, \dots, N\}$ with the weights

$$(1.3) \quad \rho(x) = \binom{N}{x} p^x q^{N-x}, \quad x = 0, 1, \dots, N.$$

It is well known that the Krawtchouk polynomials are related to the Hamming association scheme in coding theory (see, e.g. [5, 13, 15, 21]), and Lloyd's Theorem [12, 15] states that the existence of a perfect code in the Hamming metric corresponds to the Krawtchouk polynomials having integer zeros. Recently, there has been considerable interest in the asymptotics of Krawtchouk polynomials, when the degree n grows to infinity. For instance, in [20], Sharapudinov obtained the asymptotic formula

$$(1.4) \quad \begin{aligned} & (2Npq\pi n!)^{1/2} (Npq)^{-n/2} \rho(\hat{x}) e^{x^2/2} K_n^N(\hat{x}) \\ & = e^{-x^2/2} (2^n n!)^{-1/2} H_n(x) + \mathcal{O}(n^{7/4} N^{-1/2}), \end{aligned}$$

where $\hat{x} = Np + (2Npq)^{1/2}x$, $n = \mathcal{O}(N^{1/3})$, $x = \mathcal{O}(n^{1/2})$, and $H_n(x)$ is the Hermite polynomial. Furthermore, if the zeros of $K_n^N(\hat{x})$ are arranged in decreasing order: $\hat{x}_{1,N} > \hat{x}_{2,N} > \dots > \hat{x}_{n,N}$, then he has shown that

$$(1.5) \quad \hat{x}_{n,N} = Np[1 - (2q/Np)^{1/2}x_1(n)] + \mathcal{O}(n^{7/4})$$

uniformly with respect to $1 \leq n \leq \eta_N N^{1/4}$, $N = 1, 2, \dots$, where $\{\eta_N\}$ is a sequence of positive numbers tending to zero as $N \rightarrow \infty$ and $x_1(n)$ is the smallest zero of the Hermite polynomial. Also, Dragnev and Saff [6, 7] have given the distribution of the zeros of Krawtchouk polynomials.

In [9], Ismail and Simeonov have investigated the asymptotic behavior of $K_n^N(nt)$ as $n \rightarrow \infty$, when $N/n = \gamma$ is a fixed constant independent of n . They divided the t -interval $0 < t < \gamma$ into several subintervals, and gave an asymptotic formula for $K_n^N(nt)$ in each of the subintervals by using the classical saddle point method [22]. Their formulae hold uniformly on compact subsets of each subinterval, but are not valid for $x = nt$ in any bounded interval in $[0, +\infty)$ since in this case t has to tend to zero as $n \rightarrow \infty$. When $\gamma \geq 1/p$, they have also proved that the zeros of $K_n^N(nt)$ are in the interval

$$(1.6) \quad \left(p\gamma - p + q - \sqrt{pq(\gamma - 1)}, p\gamma - p + q\sqrt{pq(\gamma - 1)} \right).$$

More accurate bounds for the zeros have been given by Krasikov [10].

Recently, Li and Wong [14] have studied the uniform asymptotic behavior of $K_n^N(x)$ in the interval $0 < x < N$, as $n \rightarrow \infty$. With $\mu = n/N$ and $x = \lambda N$, they have derived an infinite asymptotic expansion for $K_n^N(\lambda N)$ as $n \rightarrow \infty$, which holds uniformly for μ and λ in any compact subinterval of $(0, 1)$. For fixed μ , their result covers the various asymptotic approximations given by Ismail and Simeonov [9]. Another treatment of uniform asymptotics of $K_n^N(x)$ with scaled variable $x = nt$ can be found in Baik, Kriecherbauer, McLaughlin and

Miller [3], where a steepest descent method for a Riemann-Hilbert problem is used. However, the results in both [9] and [3] are not valid for fixed or bounded x .

The purpose of this paper is to present an asymptotic expansion for $K_n^N(x)$ with fixed or bounded x , which is uniformly valid for n/N in $(0, 1]$. This expansion is given in terms of the confluent hypergeometric functions. To state the result more precisely, we let $\mu = N/n$. When $n \rightarrow \infty$, we have the asymptotic expansion

$$(1.7) \quad K_n^N(x) \sim (-1)^{n+1} p^{n-x} e^{-n\gamma} \cdot \left[\mathbf{N}(-n(\mu-1), x - n(\mu-1) + 1, n\eta) \sum_{k=0}^{\infty} (-1)^k \frac{a_k}{(n\eta)^k} + \mathbf{N}'(-n(\mu-1), x - n(\mu-1) + 1, n\eta) \sum_{k=0}^{\infty} (-1)^k \frac{b_k}{(n\eta)^k} \right],$$

which holds uniformly for $1 \leq \mu < \infty$ and for x in any bounded subinterval of $[0, \infty)$, where \mathbf{N} denotes the confluent hypergeometric function given in [18, p. 255], γ and η are analytic functions of μ given in Section 3 below, and a_k and b_k can be calculated recursively; see (4.16)–(4.18).

Asymptotic approximations are also obtained for the zeros of Krawtchouk polynomials $K_n^{\mu}(x; p, q)$ in various cases depending on the values of p , q and μ . Let $x_{n,k}$ denote the k -th zero of $K_n^{\mu}(x)$, $k = 0, 1, 2, \dots, n-1$, counted in increasing order

$$0 \leq x_{n,0} < x_{n,1} < \dots < x_{n,n-1} \leq N.$$

As $n \rightarrow \infty$ we have

- (i) if $n(\mu - 1/p)^2 \rightarrow \infty$ and $\mu > 1/p$ then each $x_{n,k}$ tends to infinity,
- (ii) if $n(\mu - 1/p)^2 \rightarrow \infty$ and $\mu < 1/p$ then $x_{n,k}$ equals k up to an exponentially small error;
- (iii) if $n(\mu - 1/p)^2$ is bounded then $x_{n,k}$ can be expressed in terms of the k -th zero of the parabolic cylinder function given in (4.29) below.

The precise statement of the result is given in Theorem 2 at the end of the paper.

2. The steepest descent method

By Cauchy's formula, we have from (1.1)

$$(2.1) \quad K_n^N(x) = \frac{1}{2\pi i} \int_C (1 - pw)^{N-x} (1 + qw)^x \frac{dw}{w^{n+1}},$$

where C is a small, positively oriented, closed contour surrounding $w = 0$. Changing the variable in (2.1) to $t = 1/(1 - pw)$ we have

$$(2.2) \quad K_n^N(x) = -\frac{p^{n-x}}{2\pi i} \int_{C'} \frac{(t - q)^x}{t^{N-n+1}(t - 1)^{n+1}} dt,$$

where C' is a circle surrounding $t = 1$, oriented in the negative direction and with a radius less than 1.

If we set $\mu = N/n$, then $1 \leq \mu < +\infty$ and (2.2) can be written in the form

$$(2.3) \quad K_n^{n\mu}(x) = (-1)^{n+1} \frac{p^{n-x}}{2\pi i} \int_{C'} \frac{(t-q)^x}{t(t-1)} e^{-nf(t,\mu)} dt,$$

where the phase function $f(t, \mu)$ is given by

$$(2.4) \quad f(t, \mu) = \log(1-t) + (\mu-1) \log t.$$

When $\mu > 1$, the function $f(t, \mu)$ has two branch points $t = 0$ and $t = 1$. This function is analytic and single-valued in the t -plane with cuts along the intervals $(-\infty, 0]$ and $[1, \infty)$. For all logarithmic functions in (2.4), we choose the principal branch

$$\log \zeta = \log |\zeta| + i \arg \zeta, \quad -\pi < \arg \zeta < \pi.$$

For later discussion, we also need to specify the values of the argument along the upper and lower edges of the cuts. If $t^\pm = u + i0^\pm$, $u < 0$, denote points on the upper and lower edges of the cut along $(-\infty, 0]$ respectively, then we choose

$$(2.5) \quad \arg t^+ = \pi, \quad \arg t^- = -\pi;$$

if $t^\pm = u + i0^\pm$, $u > 1$, are points on the upper and lower edges of the cut along $[1, \infty)$ respectively, we choose

$$(2.6) \quad \arg(1-t^+) = -\pi, \quad \arg(1-t^-) = \pi.$$

When $\mu = 1$, the function $f(t, \mu)$ has only one branch point $t = 1$. By choosing the principal value of the logarithmic function indicated above, $f(t, \mu)$ is analytic in the cut plane $\mathbb{C} \setminus [1, \infty)$.

The amplitude function in the integral in (2.3) also has a branch point $t = q$ when x is not an integer. For definiteness of $(t-q)^x$, we choose a cut along $(-\infty, q]$ in the t -plane and take the branch of $(t-q)^x$ in the cut plane so that $(t-q)^x$ is real when $t > q$. Thus, if $t^\pm = u + i0^\pm$, $u < q$, are points along the upper and lower edges of the cut along $(-\infty, q]$, then

$$(2.7) \quad \arg(t^\pm - q)^x = \pm \pi x.$$

The derivative of $f(t, \mu)$ with respect to t is given by

$$(2.8) \quad f'(t, \mu) = \frac{\mu t - (\mu - 1)}{t(t-1)}.$$

For $1 < \mu < +\infty$, the function $f(t, \mu)$ has a saddle point at $t_0 = (\mu - 1)/\mu$. This saddle point lies in $0 < t_0 < 1$, and approaches 0, as $\mu \rightarrow 1$, which is one of the branch points of $f(t, \mu)$. Note that the saddle point will disappear when μ reaches 1; i.e. so the function $f(t, \mu)$ has no saddle point for $\mu = 1$.

At the saddle point t_0 , we have

$$(2.9) \quad f(t_0, \mu) = (\mu - 1) \log(\mu - 1) - \mu \log \mu,$$

and

$$(2.10) \quad f''(t_0, \mu) = \frac{\mu^3}{1 - \mu} < 0.$$

If x is not an integer, there is also a cut along $(-\infty, q]$ for the amplitude function. When $t_0 < q$, or equivalently $\mu < 1/p$, we let t_0^+ and t_0^- denote the point t_0 on the upper and lower edges of the cut respectively. By taking branches as in (2.5), (2.6) and (2.7), we have

$$f(t_0^+, \mu) = f(t_0^-, \mu)$$

but

$$(t_0^+ - q)^x = \overline{(t_0^- - q)^x} = (q - t_0)^x e^{i\pi x}.$$

The relevant path of steepest descent L for the integral (2.3) is given by

$$(2.11) \quad \operatorname{Im}(f(t, \mu) - f(t_0, \mu)) = \arg(1 - t) + (\mu - 1) \arg t = 0$$

and

$$(2.12) \quad \operatorname{Re}(f(t, \mu) - f(t_0, \mu)) \geq 0.$$

It is obvious that the path of steepest descent L is symmetric with respect to the real axis. Note that points in the interval $(0, 1)$ on the real axis satisfy (2.11), but do not satisfy (2.12). Indeed, we have $\operatorname{Re}(f(t, \mu) - f(t_0, \mu)) \rightarrow -\infty$ as $t \rightarrow 0$ and $t \rightarrow 1$. Hence, the path of steepest descent L must be the path through t_0 , perpendicular to the real axis at t_0 and going to infinity; see Figure 1. If we set $t = u + iv$, $v > 0$, then the half branch L_+ of L in the upper half plane is described by the equation

$$(2.13) \quad -\operatorname{arccot} \frac{1 - u}{v} + (\mu - 1) \operatorname{arccot} \frac{u}{v} = 0.$$

The integration contour C' in (2.3) can be deformed into the oriented curve Γ shown in Figure 1. When $\mu \geq 1/p$, i.e. $t_0 \geq q$, Γ is exactly the path of steepest descent L . When $1 < \mu < 1/p$, i.e. $0 < t_0 < q$, Γ consists of the lower half branch L_- of L , the lower edge ℓ_- from t_0^- to q and the upper edge ℓ_+ from q to t_0^+ of the cut along $(-\infty, q]$, and the upper half branch L_+ of L ; see Figure 1 (b). When $\mu = 1$, Γ consists of the lower and upper edges of the cut along $(-\infty, q]$, with two small semicircular indentations near 0, since 0 is a simple pole of the integrand in (2.3); see Figure 1 (c).

The integral (2.3) can now be written as

$$(2.14) \quad K_n^{\mu}(x) = (-1)^{n+1} p^{n-x} I(n, \mu; x),$$

where

$$(2.15) \quad I(n, \mu; x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(t - q)^x}{t(t - 1)} e^{-nf(t, \mu)} dt.$$

We need consider only the integral $I(n, \mu; x)$.

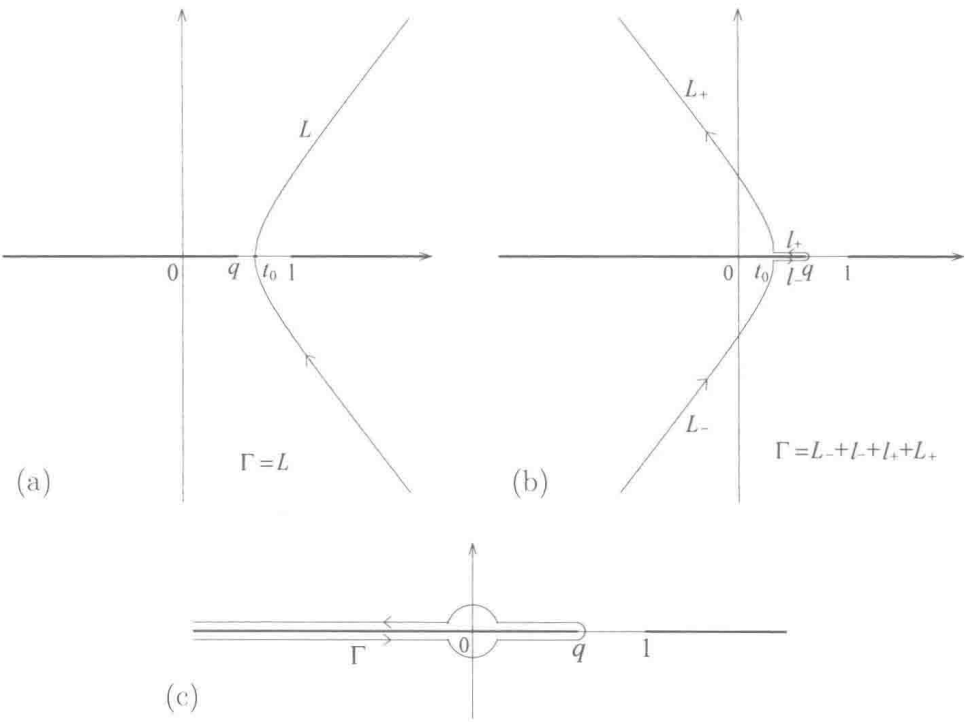


FIGURE 1. Path of steepest descent Γ : (a) $t_0 > q$; (b) $t_0 \leq q$; (c) $t_0 = 0$.

3. Relation to confluent hypergeometric functions

To obtain the asymptotic expansion of the integral (2.15) for bounded $x \in [0, +\infty)$ which holds uniformly for $1 \leq \mu < \infty$, we should compare (2.15) with a well-known special function. Note that the function $f(t, \mu)$ in (2.15) has a saddle point at t_0 . This saddle point moves close to $t = q$, the branch point of the amplitude function, when μ gets close to $1/p$; it moves close to $t = 0$, the logarithmic singularity of $f(t, \mu)$, when μ gets close to 1. To find a function having the same properties as $f(t, \mu)$, we introduce the function

(3.1)
$$\begin{aligned} \mathbf{N}(a, c; z) &= \frac{1}{2\pi i} \int_{-\infty}^{(1+)} s^{a-1} (s-1)^{c-a-1} e^{zs} ds \\ &= \frac{1}{2\pi i} \int_{-\infty}^{(1+)} (s-1)^{c-a-1} e^{zs+(a-1)\log s} ds, \end{aligned}$$

where the integration path begins at $-\infty$, encircles the point $t = 1$ once in the positive direction, and then returns to $-\infty$; see Figure 2. The cut in the s -plane is taken along $(-\infty, 1]$, and we take the principal branches for the functions s^{a-1} and $(s-1)^{c-a-1}$ in the cut plane.