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PREFACE

The second volume of *Vibro-Acoustics* includes eight chapters. As in the first volume, each chapter ends with a number of problems. The solutions are given in a separate volume, which also contains a summary of some of the most important governing equations from the first two volumes.

A number of measurement results are presented in Volume II. Most of these measurements could not have been performed without the expert help of Fritiof Torstensson and Arne Jagenäs at Chalmers, Knut Ulvund at DNV and Kent Lindgren and Danilo Prilovic at KTH. I also gratefully acknowledge the pioneering work by Prof T.Kihlman, who introduced the field of vibro-acoustics in Scandinavia.

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Any comments or questions on the text are most welcome. Contact us by email, address andersc.nilsson@gmail.com and lïubl@mail.ioa.ac.cn.

Anders Nilsson

Genova, Italy, March 2013

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Chapter 9

HAMILTON'S PRINCIPLE AND SOME OTHER VARIATIONAL METHODS

Many problems in mathematical physics and thus in vibro-acoustics cannot be solved exactly. However, a variational technique can often be used to sufficiently well formulate the equations governing the response of a structure excited by external forces. The technique ensures that errors are minimized. Variational techniques are excellent tools for solving dynamic problems for which exact solutions cannot be formulated.

The widely used Finite Element Method, is based on Hamilton's principle, which is a very powerful variational method. The principle can be proved based on Newton's law of motion. Inversely Newton's law can be derived using Hamilton's principle. However, Hamilton's principle is much more general than Newton's law and for this reason, it has survived the revolution in mechanics brought by Einstein.

The key problem for the successful application of any variational technique is the mathematical formulation of the kinetic and potential energies of a system. This formulation also requires a physical understanding of the mechanisms governing the motion of a system. This can be illustrated by considering two different types of three layered beams. In one case, the structure consists of a beam with a constrained viscoelastic layer. For the vibrating beam, the shear forces in the viscoelastic layer along the axis of the beam are of importance. In the other case, the core of a three-layered beam consists of a honeycomb structure. In this case, the shear forces perpendicular to the axis of the beam are of major importance for the deflection of the beam. For the two cases, the energies are modelled in different ways resulting in two different equations as discussed in Sections 9.3 and 9.4. In each case, the results are only valid as long as the basic physical assumptions are satisfied.

Hamilton's principle is in this chapter used to derive the equations, which up to certain frequencies govern the flexural vibrations of thick beams or plates and of cylindrical shells. The Lagrange and Garlekin methods are also discussed. The longitudinal vibration of thick beams or rods is examined in Chapter 10 in connection with discussions on various models describing the axial vibration of cylindrical rubber mounts.

9.1 Hamilton's principle

The most general formulation of the law governing the motion of a mechanical system is Hamilton's principle. In formulating the principle, it is assumed that the potential energy $\mathcal{V}$ is a known function of some generalized coordinates $q_1, q_2, \cdots, q_n$ and time t. The potential energy is symbolically written as $\mathcal{V} = \mathcal{V}(q, t)$. The kinetic energy $\mathcal{T}$ of the same system is also assumed to be a known function of the coordinates $q_1, q_2, \cdots, q_n$, the velocities $\dot{q}_1, \dot{q}_2, \cdots, q_n$ and time t. Thus, the kinetic energy is written as $\mathcal{T} = \mathcal{T}(q, \dot{q}, t)$.

The Hamilton's principle states: between two instants of time, t_1 and t_2, the motion of a mechanical system is such that for the coordinates defining the system to be described by the functions $q_i(t)$ the integral

$$\mathcal{J} = \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) \mathrm{d}t \tag{9-1}$$

is stationary. It assumed that the coordinates or displacements of the system at $t = t_1$ and $t = t_2$ are known.

Hamilton's principle can also be written in the form

$$\delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V}) \, \mathrm{d}t = 0 \tag{9-2}$$

The expression states that the variation of the integral is zero when the system is given a virtual displacement if the virtual displacement is zero at $t = t_1$ and $t = t_2$. During time period t_1 to t_2 the system will move in such a way that the time average of the difference between the kinetic and potential energies is an extrenum or in most cases a minimum.

The difference between the kinetic and potential energies is called the Lagrangian of the system and is defined as (or sometimes as $-\mathcal{L}$)

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \tag{9-3}$$

The influence of an external field or force can also be incorporated in the variational expression. By defining the potential energy for the conservative external forces as $\mathcal{A}$ and by including this in the original expression (9-2) Hamilton's principle reads

$$\delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V} - \mathcal{A})\mathrm{d}t = 0 \tag{9-4}$$

For a conservative (no losses) and external force described by the vector $\boldsymbol{F}$ acting on a particle and moving the particle along a path given by the vector $\boldsymbol{s}$ the potential energy of the external force is reduced and giving the energy as

$$\mathcal{A} = -\int \boldsymbol{F}\mathrm{d}\boldsymbol{s} \tag{9-5}$$

Hamilton's principle as a tool for deriving the governing equations describing the motion of a simple beam and some more complicated structures are discussed in the following sections.

One proof of Hamilton's principle can, as suggested by Petyt in ref. [63], be illustrated by considering a simple system shown in Fig. 9-1. Let the vector $\boldsymbol{F}$ describe a conservative force acting on the point mass m. The work $\mathcal{W}$ done by a conservative force, defined by the vector $\boldsymbol{F}$, when moving the mass m from a position $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ is independent of the path taken. The work $\mathcal{W}$ done along any path $\boldsymbol{s}$ is

$$\mathcal{W} = \int \boldsymbol{F}\mathrm{d}\boldsymbol{s}$$

During the process the mass or the system has lost the potential energy $\mathcal{V}$, thus $\mathcal{V} = -\mathcal{W}$. The conservative vector force $\boldsymbol{F}$ can thus be expressed as a

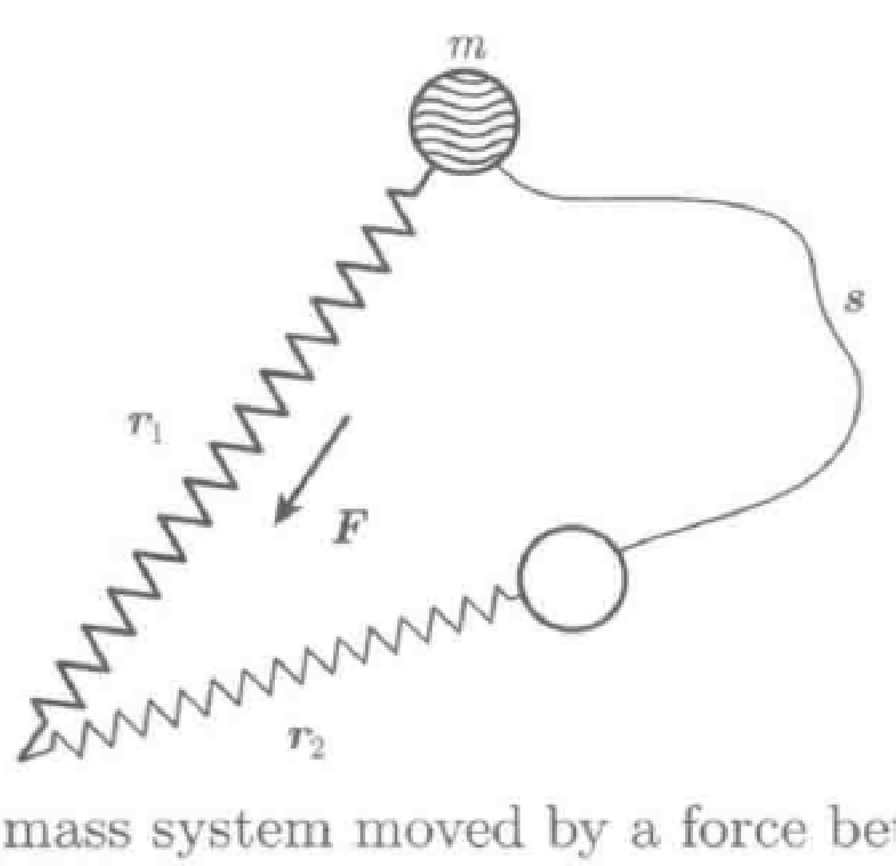

Fig. 9-1 A simple mass system moved by a force between two positions

function of the potential energy $\mathcal{V}$ of the symbol

$$\boldsymbol{F} = -\mathbf{grad}\ \mathcal{V} \tag{9-6}$$

Consider now a simple mass m subjected to a conservative force defined by the vector $\boldsymbol{F}$. According to the principle of virtual displacement, it follows that

$$\boldsymbol{F} \cdot \delta\boldsymbol{r} - m\ddot{\boldsymbol{r}} \cdot \delta\boldsymbol{r} = 0 \tag{9-7}$$

The vector $\boldsymbol{r}$ defines the position of the mass. The virtual work $\delta\mathcal{W} = \boldsymbol{F} \cdot \delta\boldsymbol{r}$ done by the conservative force is also given by $\delta\mathcal{V} = -\boldsymbol{F} \cdot \delta\boldsymbol{r}$ when $\mathcal{V}$ is the potential energy of the system. Considering this, eq. (9-7) is written as

$$\delta\mathcal{V} = -m\ddot{\boldsymbol{r}} \cdot \delta\boldsymbol{r} \tag{9-8}$$

The kinetic energy $\mathcal{T}$ of the mass or in fact the system is $\mathcal{T} = m \cdot \dot{\boldsymbol{r}}^2/2$. Thus

$$\delta\mathcal{T} = m \cdot \dot{\boldsymbol{r}} \cdot \delta\dot{\boldsymbol{r}} \tag{9-9}$$

By subtracting the expression (9-8) from (9-9) the result is

$$\delta\mathcal{T} - \delta\mathcal{V} = m \cdot \dot{\boldsymbol{r}} \cdot \delta\dot{\boldsymbol{r}} + m\ddot{\boldsymbol{r}} \cdot \delta\boldsymbol{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\boldsymbol{r}} \cdot \delta\boldsymbol{r}\right) \tag{9-10}$$

Integration with respect to time from t_1 to t_2 gives

$$\int_{t_1}^{t_2} (\delta\mathcal{T} - \delta\mathcal{V})\,\mathrm{d}t = [m\dot{\boldsymbol{r}} \cdot \delta\boldsymbol{r}]_{t_1}^{t_2} \tag{9-11}$$

Assuming that the virtual displacement $\delta\boldsymbol{r}$ is equal to zero for $t = t_1$ and $t = t_2$ it follows that

$$\delta \int_{t_1}^{t_2} (\mathcal{T} - \mathcal{V})\,\mathrm{d}t = 0$$

This is Hamilton's result as given by eq. (9-2).

Hamilton's principle is discussed in for example refs. [58], [77] and [78].

9.2 Flexural vibrations of slender beams

The flexural vibrations of "thin" or slender beams were discussed in Chapter 7(Volume I). Again, a "thin" beam under flexure is considered to demonstrate how Hamilton's principle can be used to derive the equations governing the motion of the beam as well as to formulate the boundary conditions of the beam.

A simple homogeneous and slender beam is shown in Fig. 9-2. The bending stiffness of the beam is D' and it's mass per unit length m'. The beam is extended along the x-axis from $x = 0$ to $x = L$ and is excited by a force $F'(x, t)$ per unit length. The resulting displacement of the beam is $w(x, t)$ defined positive, as the force, along the positive y-axis. The forces and bending moments at the boundaries are F_1, F_2, M_1 and M_2 as defined in Fig. 9-2.

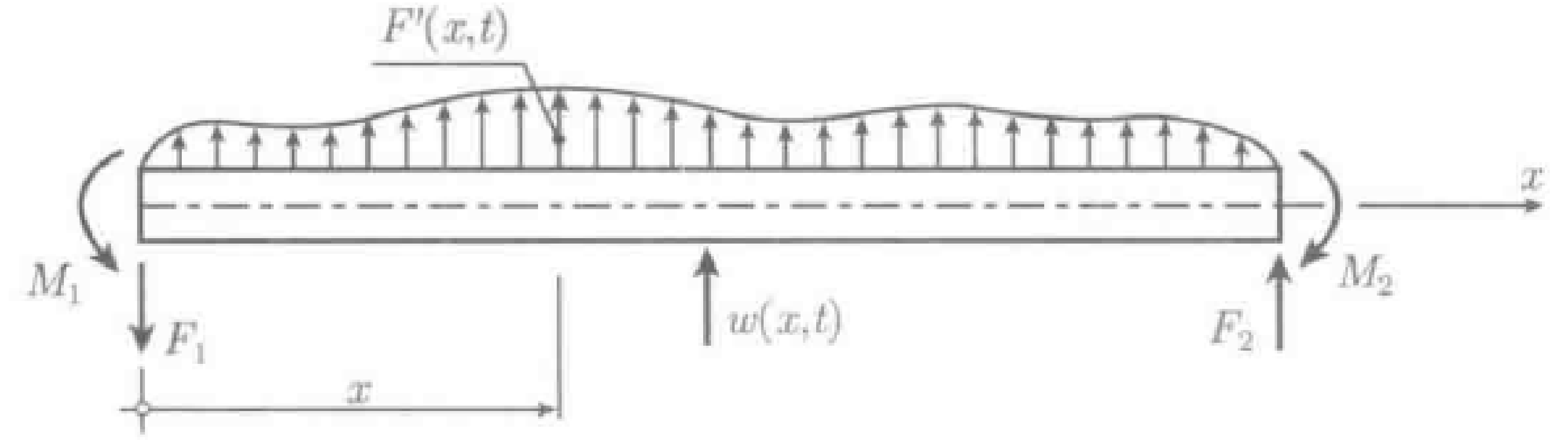

Fig. 9-2 Forces and bending moments acting on a beam

The potential energy $\mathcal{U}_l$ induced by bending is per unit length of the beam given by eq. (3-84) as

$$\mathcal{U}_l = \frac{D'}{2} \cdot \left(\frac{\partial^2 w}{\partial x^2}\right)^2$$

The kinetic energy per unit length of the beam is

$$\mathcal{T}_l = \frac{m'}{2} \cdot \left(\frac{\partial w}{\partial t}\right)^2$$

The potential energy $\mathcal{A}_1$ per unit length induced by the external force F' is according to the definition (9-5) given by

$$\mathcal{A}_1(t) = -\int_0^L w(x,t)F'(x,t)\mathrm{d}x$$

The potential energy $\mathcal{A}_2$ induced by the external forces and moments is

$$\begin{aligned}\mathcal{A}_2(t) &= -F_2(t)w(L,t) + F_1(t)w(0,t) + M_2(t)\left[\frac{\partial w}{\partial x}\right]_{x=L} - M_1(t)\left[\frac{\partial w}{\partial x}\right]_{x=0} \\ &= -\left[Fw - M \cdot \frac{\partial w}{\partial x}\right]_0^L \qquad (9\text{-}12)\end{aligned}$$

The total potential energy invoked by all external forces is consequently

$$\mathcal{A}(t) = -\int_0^L w(x,t)F'(x,t)\mathrm{d}x - \left[Fw - M \cdot \frac{\partial w}{\partial x}\right]_0^L \qquad (9\text{-}13)$$

According to Hamilton's principle the kinetic and potential energies should satisfy by inserting eqs. (9-4), (9-5). Thus

$$\delta\int_{t_1}^{t_2}(\mathcal{T}-\mathcal{U}-\mathcal{A})\mathrm{d}t=\delta\int_{t_1}^{t_2}\mathrm{d}t\left[-\mathcal{A}+\int_0^L\mathrm{d}x(\mathcal{T}_l-U_l)\right]=0 \tag{9-14}$$

By inserting eqs. (9-4), (9-5) and (9-12) in eq. (9-13) the result is

$$\begin{aligned}&\delta\int_{t_1}^{t_2}\mathrm{d}t\left[\int_0^L\mathrm{d}x\left\{\frac{m'}{2}\cdot\left(\frac{\partial w}{\partial t}\right)^2-\frac{D'}{2}\cdot\left(\frac{\partial^2 w}{\partial x^2}\right)^2+wF'\right\}\right]\\&+\left[Fw-M\cdot\frac{\partial w}{\partial x}\right]_0^L=0\end{aligned} \tag{9-15}$$

Effecting the variation the expression (9-15) is written as

$$\begin{aligned}&\int_{t_1}^{t_2}\mathrm{d}t\left[\int_0^L\mathrm{d}x\left\{m'\cdot\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial\delta w}{\partial t}\right)-D'\cdot\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2\delta w}{\partial x^2}\right)+F'\delta w\right\}\right]\\&+\int_{t_1}^{t_2}\mathrm{d}t\left[F\delta w-M\cdot\frac{\partial\delta w}{\partial x}\right]_0^L=0\end{aligned} \tag{9-16}$$

Next, integration by parts is carried out. For simplicity, each part of the integrand is treated separately. The first part of the double integral of (9-16) gives

$$X_1=\int_0^L\mathrm{d}x\int_{t_1}^{t_2}\mathrm{d}t\cdot m'\frac{\partial w}{\partial t}\frac{\partial\delta w}{\partial t}=\int_0^L\mathrm{d}x\left\{\left[m'\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\mathrm{d}t\cdot m'\frac{\partial^2 w}{\partial t^2}\delta w\right\}$$

However, as required, the displacement w is fixed at the initial and final time limits. Thus $\delta w=0$ at $t=t_1$ and $t=t_2$. The integral X_1 is consequently reduced to

$$X_1=-\int_{t_1}^{t_2}\mathrm{d}t\cdot m'\frac{\partial^2 w}{\partial t^2}\delta w \tag{9-17}$$

The second part X_2 of the integral (9-16) is integrated by parts as

$$\begin{aligned}X_2&=\int_{t_1}^{t_2}\mathrm{d}t\left\{\int_0^L\mathrm{d}x\left[-D'\cdot\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2\delta w}{\partial x^2}\right)\right]\right\}\\&=-D'\int_{t_1}^{t_2}\mathrm{d}t\Bigg[\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial\delta w}{\partial x}\right)\end{aligned}$$

$$- \left(\frac{\partial^3 w}{\partial x^3} \right) \delta w \Big]_0^L - D' \int_0^L \mathrm{d}x \int_{t_1}^{t_2} \mathrm{d}t \cdot \frac{\partial^4 w}{\partial x^4} \cdot \delta w \tag{9-18}$$

By introducing the expressions (9-17) and (9-18) in eq. (9-16) the result is

$$\int_{t_1}^{t_2} \mathrm{d}t \int_0^L \mathrm{d}x \delta w \left[-m' \frac{\partial^2 w}{\partial t^2} - D' \frac{\partial^4 w}{\partial x^4} + F' \right] + \int_{t_1}^{t_2} \mathrm{d}t \left[\delta w \left(D' \frac{\partial^3 w}{\partial x^3} + F \right) - \frac{\partial \delta w}{\partial x} \left(D' \frac{\partial^2 w}{\partial x^2} + M \right) \right]_0^L = 0 \tag{9-19}$$

For the result to be zero for any δw or $\partial \delta w / \partial x$, it follows that the expressions inside the brackets must be zero. Setting the expression inside the first bracket equal to zero gives

$$D' \frac{\partial^4 w}{\partial x^4} + m' \frac{\partial^2 w}{\partial t^2} = F' \tag{9-20}$$

This is the equation of motion of a slender and homogeneous beam in flexure as already discussed in Section 3.7.

By setting the second square bracket equal zero, the boundary conditions for $x = 0$ and $x = L$ are obtained as

$$\delta w \left(D' \frac{\partial^3 w}{\partial x^3} + F \right) = 0 \tag{9-21}$$

$$\frac{\partial \delta w}{\partial x} \left(D' \frac{\partial^2 w}{\partial x^2} + M \right) = 0 \tag{9-22}$$

For the first condition (9-21) to be satisfied at a boundary, it follows that

$$F = -D' \frac{\partial^3 w}{\partial x^3} \quad \text{or} \quad \delta w = 0 \tag{9-23}$$

The second condition requires

$$M = -D' \frac{\partial^2 w}{\partial x^2} \quad \text{or} \quad \frac{\partial \delta w}{\partial x} = 0 \tag{9-24}$$

The results give the expressions for force and bending moment as given by the displacement w of the beam as already derived in Chapter 3, eqs.(3-73) and (3-75). The conditions $\delta w = 0$ and $\delta(\partial w / \partial x) = 0$ are equivalent to w and $\partial w / \partial x$ being constant at the boundaries. For most practical purposes a coordinate system can be oriented in such a way that the boundary conditions can be written $w = 0$ and $\partial w / \partial x = 0$.