New Syllabus Mathematics for O-Level 2

Owen and Joyce Perry

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Owen Perry, B.Sc.,
Head of Department of General and Professional Studies, Lewes Technical College

Joyce Perry, B.Sc.,
Formerly Mathematics Teacher, Lewes High School

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Preface

These two volumes are intended for students who want to pass O-Level mathematics in the modern syllabus. They are particularly suitable for those who need to follow a thorough revision course, whether at school or as full-time, day-release or evening students at colleges of further education. Since the only mathematical knowledge assumed is simple arithmetic, the books are also suitable for those who need a pass in O-Level mathematics to improve their promotion prospects, and are starting the modern syllabus for the first time.

The majority of the exercises are divided into A and B sections. The questions in the A sections are generally shorter and intended for routine practice in the techniques appropriate to each part of the text. Longer and more thought-provoking questions are found in the B sections. Each of the sixteen chapters ends with a multiple-choice test and a selection of miscellaneous examples from past examination papers.

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The text covers the 'modern' alternative syllabus of each of the major examining boards, and the authors acknowledge with thanks the permission given by these boards to quote examination questions. The source of each question is shown in the text by the following abbreviations

(AEB)	Associated Examining Board
(C)	University of Cambridge Local Examinations Syndicate
(L)	University of London, University Entrance and School Examinations Council
(JMB)	Joint Matriculation Board
(NI)	Northern Ireland Schools Examinations Council
(O)	Oxford Delegacy of Local Examinations
(OC SMP)	Oxford and Cambridge Schools Examination Board. Schools Mathematics Project
(S)	Southern Universities Joint Board
(SCE)	Scottish Certificate of Education Examination Board.

Notation

$\{a, b, c, \ldots\}$	the set of $a, b, c, \ldots$
$:$	such that
$\in$	is an element of
$\notin$	is not an element of
$n(\ )$	the number of elements in the set of
$\varnothing$	the empty (null) set
$\mathscr{E}$	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a subset of
A'	the complement of the set A
N	the set of natural numbers
Z	the set of integers
R	the set of real numbers
PQ	operation Q followed by operation P
$f{:}x \to y$	the function of mapping the set X into the set Y
$f(x)$	the image of x under the function f
f^{-1}	the inverse of the function f
fg	the function f of the function g
—○—○—	open interval on the number line
—●—●—	closed interval on the number line
$\{x: -2 < x < 7\}$	the set of values of x such that . . .
$\Rightarrow$	implies that
$\Leftarrow$	is implied by
$\Leftrightarrow$	implies and is implied by
$=$	is equal to
$\equiv$	is identically equal to
$\approx$	is approximately equal to
$\neq$	is not equal to
$<$	is less than
$\leqslant$	is less than or equal to
$>$	is greater than
$\geqslant$	is greater than or equal to
$\nless$	is not less than
$\ngtr$	is not greater than
$\lvert\ \rvert$	the unsigned part of a signed number, that is the modulus
∞	infinity
M'	the transpose of the matrix M

Contents

1 Quadratic Functions

Chapter 8 of volume 1 was concerned mainly with linear functions and the equations and inequalities derived from them. We shall now consider functions containing higher powers.

1.1 Polynomial Functions

A function such as $f(x) = a + a_1x + a_2x^2 + a_3x^3 + \ldots a_nx^n$, where n is the highest power of x and $a, a_1, a_2 \ldots a_n$ are real constants, is called a polynomial function of degree n.

$x^5 + 3x^2 + 2x + 7$ is a polynomial of degree 5 in x. A linear function $ax + b$ is of the first degree, and is called a binomial since there are two terms. The function $f(x) = ax^2 + bx + c$ is of the second degree in x and is a trinomial, but since it could be obtained by multiplying two binomials together, it is usually called a quadratic function.

For instance, using the distributive law

$$\begin{aligned}(2x+3)(x+1) &= 2x(x+1) + 3(x+1)\\ &= 2x^2 + 2x + 3x + 3\\ &= 2x^2 + 5x + 3\end{aligned}$$

Similarly the product of three linear functions is a polynomial of degree three and in general the product of K linear functions of x is a polynomial of degree K in x.

Example 1.1

Express the following products as polynomials
(a) $(2x-3)(x-7)$ (b) $(x+3)(2x-3)(x-7)$
(c) $(ax+b)^2$ (d) $(ax-b)^2$ (e) $(ax+b)(ax-b)$.

(a) $$\begin{aligned}(2x-3)(x-7) &= 2x(x-7) - 3(x-7)\\ &= 2x^2 - 14x - 3x + 21\\ &= \underline{2x^2 - 17x + 21}\end{aligned}$$

(b) $$\begin{aligned}(x+3)(2x-3)(x-7) &= (x+3)(2x^2-17x+21)\\ &= x(2x^2-17x+21)\\ &\quad + 3(2x^2-17x+21)\\ &= 2x^3 - 17x^2 + 21x\\ &\quad + 6x^2 - 51x + 63\\ &= \underline{2x^3 - 11x^2 - 30x + 63}\end{aligned}$$

(c) $$\begin{aligned}(ax+b)^2 &= (ax+b)(ax+b)\\ &= ax(ax+b) + b(ax+b)\\ &= a^2x^2 + abx + bax + b^2\\ &= \underline{(ax)^2 + 2abx + b^2}\end{aligned}$$

(d) $$\begin{aligned}(ax-b)^2 &= (ax-b)(ax-b)\\ &= ax(ax-b) - b(ax-b)\\ &= a^2x^2 - abx - bax + b^2\\ &= \underline{(ax)^2 - 2abx + b^2}\end{aligned}$$

(e) $$\begin{aligned}(ax+b)(ax-b) &= ax(ax-b) + b(ax-b)\\ &= a^2x^2 - abx + bax - b^2\\ &= \underline{(ax)^2 - b^2}\end{aligned}$$

These last three results should be studied carefully as they are very important, and the last one, called the difference of two squares, is used a great deal in later work.

1.2 Factorisation

The converse statement 'any polynomial of degree K in x can be expressed as a product of K linear factors' is also true, but rather more difficult to verify except in simple cases. The process of converting a polynomial into a product is called *factorisation* and is the reverse of removing brackets by use of the distributive law.

Example 1.2

Factorise

(a) $8x + 4x^2 - 2x^3$

The factor common to all three terms is $2x$.

$8x + 4x^2 - 2x^3 = \underline{2x(4 + 2x - x^2)}$

(b) $3rst^2 - 9st^3 + 6r^2s^2t^4$

Here the common factors are 3, s, t^2, and expressed as a product the expression becomes

$\underline{3st^2(r - 3t + 2r^2st^2)}$

Example 1.3

Express as a product of two factors

(a) $ab + ac + bd + cd$

The terms are grouped in pairs and a common factor taken from each pair.

$(ab + ac) + (bd + cd) = a(b + c) + d(b + c)$

By the reverse of the distributive law

$a(b + c) + d(b + c) = \underline{(a + d)(b + c)}$

(b) $2x - 4 - xy + 2y$

$$(2x - 4) - (xy - 2y) = 2(x - 2) - y(x - 2) = \underline{(2 - y)(x - 2)}$$

(c) $m^3 - m^2 + m - 1$

Here the first two terms contain a common factor m^2, but the last two terms have only 1 in common. (1 is a factor of every term since it is the identity for multiplication.)

$$m^3 - m^2 + m - 1 = m^2(m - 1) + 1(m - 1) = \underline{(m^2 + 1)(m - 1)}$$

(d) $t^5 + 6 - 2t^3 - 3t^2$

Since 6 has no factor in common with t^5, but has a common factor with both $2t^3$ and $3t^2$, it is better to rearrange the expression to group the terms in pairs.

$t^5 - 3t^2 - 2t^3 + 6 = t^2(t^3 - 3) - 2(t^3 - 3)$

and taking out the common factor we have $\underline{(t^3 - 3)(t^2 - 2)}$.

Factors of Quadratic Functions

Every quadratic function with real coefficients can be expressed as the product of two linear functions and when the linear functions have integer coefficients they can sometimes be found by factorising. Finding the factors is a matter of trial and error but there are general rules to follow and after sufficient practice the correct factors are usually found quite easily.

(1) *When the coefficient of x is 0 and the constant term is negative.* In this case the two linear factors have the same terms, one is their sum and the other the difference.

$x^2 - 9 = (x + 3)(x - 3)$; $36 - 9x^2 = 9(2 + x)(2 - x)$

$4x^2 - 25 = (2x + 5)(2x - 5)$

$17 - 2x^2 = (\sqrt{17} + \sqrt{2}x)(\sqrt{17} - \sqrt{2}x)$

This type of quadratic function is called the difference of two squares and it can be used in arithmetic as shown by the following example.

Example 1.4

(a) Calculate the exact value of $18.54^2 - 1.46^2$.

$$18.54^2 - 1.46^2 = (18.54 + 1.46)(18.54 - 1.46) = 20 \times 17.08 = \underline{341.6}$$

(b) In binary arithmetic, calculate $11101^2 - 1101^2$.

$$11101^2 - 1101^2 = (11101 + 1101)(11101 - 1101) = 101010 \times 10000 = \underline{1010100000}$$

(2) *When the coefficient of x^2 is 1.*

Look carefully at the following expressions.

$x^2 + 5x + 6 = x^2 + 2x + 3x + 6 = (x + 3)(x + 2)$

$x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = (x - 3)(x - 2)$

$x^2 + x - 6 = x^2 - 2x + 3x - 6 = (x + 3)(x - 2)$

$x^2 - x - 6 = x^2 + 2x - 3x - 6 = (x - 3)(x + 2)$

In a quadratic function the two terms of the first degree are combined to give a trinomial and so in factorising they must be separated in such a way that there is a common factor. The constant terms in each pair of brackets give the constant term in the trinomial when multiplied together and the coefficient of x when added together.

Thus $3 \times 2 = 6$ $3 + 2 = 5$

$-3 \times -2 = 6$ $-3 + -2 = -5$

$3 \times -2 = -6$ $3 - 2 = 1$

$-3 \times 2 = -6$ $-3 + 2 = -1$

The method is simply to try the pairs of factors of the constant term in a quadratic expression to be factorised until you find the pair which sum to the coefficient of x. For example

$x^2 + 7x + 6 = (x + 6)(x + 1)$

$6 \times 1 = 6$ $6 + 1 = 7$

$x^2 + 5x - 6 = (x + 6)(x - 1)$

$6 \times -1 = -6$ $6 - 1 = 5$

$x^2 + 7x + 12 = (x + 4)(x + 3)$

$3 \times 4 = 12$ $3 + 4 = 7$

$x^2 - x - 12 = (x - 4)(x + 3)$

$3 \times -4 = -12$ $3 - 4 = -1$

Note that if the constant term is positive, the sign in both brackets is the same as the sign of the term in x. When the constant term is negative the brackets have different signs, and the numerically greater factor has the sign of the term in x.

(3) *When the coefficient of x^2 or the constant term is a prime number.* For example, when factorising $2x^2-17x+21$, the coefficients of x in the two factors must be 1 and 2, and both signs are negative. Pairs of factors of the constant term are tried in a systematic manner.

$$\begin{array}{cccccc} 2x & \times & -7 & -3 & -21 & -1 \\ x & & -3 & -7 & -1 & -21 \end{array}$$

The products of the first pair give $-6x-7x \neq -17x$. The next pair give $-14x-3x = -17x$. Therefore the factors of $2x^2-17x+21$ are $\underline{(2x-3)(x-7)}$.

Example 1.5

Express $6x^2-13x-5$ as a product of two factors. In this example, the constant term is prime, and so the constant terms in the factors are 1 and 5.

$$\begin{array}{cccccccccc} 6x & \times & +1 & -1 & +5 & -5 & 3x & & +1 & -5 \\ x & & -5 & +5 & -1 & +1 & 2x & & -5 & +1 \end{array}$$

The correct pair is found to be $\underline{(3x+1)(2x-5)}$.

(4) *When neither the coefficient of x^2 nor the constant term is prime.* Facility at choosing likely combinations improves with practice. Each pair of factors is tried in turn, until the correct combination is found.

Example 1.6

Factorise the quadratic function

$6x^2-17x+12$

Since the middle term is negative, the factors of 12 must be negative, since their product is positive.

$$\begin{array}{cccccccc} 3x & \times & -6 & -2 & -3 & -4 & -12 & -1 \\ 2x & & -2 & -6 & -4 & -3 & -1 & -12 \end{array}$$

Trying the first pair $3x \times -2 + 2x \times -6 = -6x-12x = -18x$.

Repeating with successive pairs, the correct combination is

$3x \times -3 + 2x \times -4 = -17x$

Therefore the factors of $6x^2-17x+12$ are $\underline{(3x-4)(2x-3)}$.

If none of these combinations had given the correct product we would have tried $6x$ and x.

Example 1.7

Factorise (a) $8x^2-8x-30$, (b) $12x^2-27$.

(a) $8x^2-8x-30 = 2(4x^2-4x-15)$
$= \underline{2(2x-5)(2x+3)}$

(b) $12x^2-27 = 3(4x^2-9)$
$= \underline{3(2x+3)(2x-3)}$

Exercise 1.1

(1) Express the following products as polynomial functions
(a) $(x+2)(x+7)$, (b) $(x-2)(x+3)$,
(c) $(x+5)(x-1)$, (d) $(2y+3)(3y-4)$,
(e) $(3y-7)(2y-3)$, (f) $(2r-5)(7r+2)$,
(g) $(3t-6)(3t+6)$, (h) $(2x-y)(2x+y)$,
(i) $(a+b)^2$, (j) $(2x+y)^2$, (k) $(3x-2)^2$,
(l) $(x+3)(x-2)(2x+1)$,
(m) $(3y-1)(2y+1)(y-1)$.
(2) Factorise
(a) $2x^3-3x^2$, (b) r^2s+rs^2, (c) $2x+x^2+x^3y$,
(d) $3mn-6mn^2$, (e) a^2-b^2, (f) $4x^2-y^2$,
(g) $9t^2-1$, (h) $3xy^2-27x$, (i) $2x^2y^2-8t^2$.
(3) Express as a product of two factors
(a) $2x^2+3x+2xy+3y$, (b) $3xy^2-xt-12y^3+4yt$,
(c) $ax+ay-2x^2-2xy$, (d) $3x^2+2xy-6x-4y$,
(e) $2st+2s^2-2rt-2rs$.
(4) Factorise the following quadratic functions
(a) x^2+4x+3, (b) x^2-6x+5, (c) y^2+6y-7,
(d) z^2+5z+6, (e) $x^2-4x-21$, (f) m^2+2m-8,
(g) $x^2-8x+12$, (h) $x^2-2x-24$,
(5) Factorise
(a) $2x^2+5x+3$, (b) $3x^2+10x-8$,
(c) $5x^2-21x+4$, (d) $7y^2-17y+6$,
(e) $3t^2+5t-12$, (f) $4m^2+2m-12$,
(g) $6x^2-x-12$, (h) $9t^2+24t+16$,
(i) $36-36t+9t^2$, (j) $6+10x-4x^2$,
(k) $27x^2-54x+24$.
(6) Calculate the exact value of
(a) 2051^2-2049^2, (b) $12^3-12^2\times 2$,
(c) $(12.698)^2-(11.302)^2$.

1.3 Quadratic Equations and Inequalities

When a polynomial function of x takes a specific value it becomes a polynomial equation, a second degree polynomial becomes a quadratic equation, a third degree polynomial a cubic equation and so on. When the product of two factors is zero, then at least one of them must be equal to zero, since, for real numbers

$ab = 0 \Rightarrow a = 0$ or $b = 0$.

For example

$(x-b)(x-c) = 0 \Rightarrow x-b = 0, \quad x = b$

$\text{or } x-c = 0, \quad x = c$

A solution of the equation $f(x) = 0$ (a value of x which satisfies it) is called a root of the equation. Every quadratic equation has two roots, although it is not always possible to find real solutions. Equations such as $x^2+4 = 0$ have no real roots, since no real number has a square which is negative.

Example 1.8

Write down the solutions of the following equations.

(a) $(x+2)(2x-3) = 0$

$x+2 = 0 \Rightarrow x = -2,\ 2x-3 = 0 \Rightarrow x = \frac{3}{2}$

The solutions are $x = -2,\ x = \frac{3}{2}$.

(b) $(x+1)(x-1) = 0$

Either $x+1 = 0 \Rightarrow x = -1$

or $x-1 = 0 \Rightarrow x = +1$

The solutions are therefore $x = \pm 1$.

(c) $x(3x-12) = 0$

Either $x = 0$ or $3x-12 = 0$

The solutions are $x = 0,\ x = 4$.

(d) $(x-5)^2 = 0$. There is only one solution, the roots are coincident, $x = 5$.

1.4 The Solution of Quadratic Equations and Inequalities by Factorisation

Example 1.9

Factorise the quadratic function $f(x) = x^2-5x+6$ and find the solution sets (a) $f(x) = 0$, (b) $f(x) < 0$, (c) $f(x) > 0$.

$x^2-5x+6 = (x-2)(x-3)$

(a) $f(x) = 0 \Rightarrow (x-2)(x-3) = 0$

$(x-2) = 0 \Rightarrow x = 2$

$(x-3) = 0 \Rightarrow x = 3$

Therefore the solution set for $x^2-5x+6 = 0$ is $\{2, 3\}$.

(b) $f(x) < 0 \Rightarrow (x-2)(x-3) < 0$

When the product of two factors is negative, one of them must be positive and one negative.

In this example, since $(x-2)$ is greater than $(x-3)$, $(x-2)$ is the positive factor and $(x-3)$ is negative.

$x-2 > 0 \Rightarrow x > 2$

$x-3 < 0 \Rightarrow x < 3$

The solution set for $x^2-5x+6 < 0$ is $(x: 2 < x < 3)$

(c) $f(x) > 0 \Rightarrow (x-2)(x-3) > 0$

When the product of two factors is positive they must have the same sign, either both positive or both negative.

$(x-2) > 0$ and $(x-3) > 0 \Rightarrow x > 2$ and $x > 3$

If x is greater than 3 it must be greater than 2 therefore $x > 3$ is a sufficient condition.

$(x-2) < 0$ and $(x-3) < 0 \Rightarrow x < 2$ and $x < 3$

but $x < 2 \Rightarrow x < 3$

therefore $x < 2$ is a sufficient condition.

The solution set for $x^2-5x+6 > 0$ is $\{x: x < 2,\ x > 3\}$.

Example 1.10

Find the range of values of x for which $4x^2-25 \geqslant 0$

$4x^2-25 = 0 \Rightarrow (2x-5)(2x+5) = 0$

$\Rightarrow x = \frac{5}{2} \text{ or } x = \frac{-5}{2}$

$4x^2-25 > 0 \Rightarrow (2x-5)(2x+5) > 0$

The factors are both positive or both negative.

Both positive $\Rightarrow 2x-5 > 0 \Rightarrow x > \frac{5}{2}$

Both negative $\Rightarrow 2x+5 < 0 \Rightarrow x < \frac{-5}{2}$

The required range of values is $x \leqslant \frac{-5}{2},\ x \geqslant \frac{5}{2}$.

Example 1.11

For what values of x is $2x^2-17x+21 \leqslant 0$?

The factors of $2x^2-17x+21$ are $(2x-3)(x-7)$.

$(2x-3)(x-7) = 0 \Rightarrow x = \frac{3}{2} \text{ or } x = 7$

$(2x-3)(x-7) < 0 \Rightarrow$ the factors have opposite signs

$2x-3 < 0,\ x-7 > 0 \Rightarrow x < \frac{3}{2},\ x > 7$

which is not consistent

$2x-3 > 0,\ x-7 < 0 \Rightarrow x > \frac{3}{2},\ x < 7$

The values of x satisfying $2x^2-17x+21 \leqslant 0$ are in the closed interval $\frac{3}{2} \leqslant x \leqslant 7$

1.5 Solution of Quadratic Equations by the Formula

$x^2+2bx+b^2 \equiv (x+b)^2$

The symbol $\equiv$ means 'is identically equal to', and indicates that the relation holds for all values of x.

$$x^2+2bx \equiv (x+b)^2-b^2 \qquad (1)$$

This provides a method of solving quadratic equations, whether or not the function factorises easily.

Any quadratic equation can be written as

$ax^2+bx+c=0$

where a, b, and c are real constants, $a \neq 0$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$x^2+\frac{b}{a}x=-\frac{c}{a} \qquad (2)$$

Comparing with (1), L.H.S. of (2) becomes

$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2$ and this is equal to $-\frac{c}{a}$

$$\therefore \left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}=\frac{b^2-4ac}{4a^2}$$

Taking the square root of both sides of this equation leads to

$$x+\frac{b}{2a}=\pm\sqrt{\left(\frac{b^2-4ac}{4a^2}\right)}$$

$$\text{Hence } x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

This is the formula for solving quadratic equations. It should be memorised although it is sometimes provided in the list of formulae for examination candidates.

Example 1.12

Calculate the roots of the following equations correct to 2 decimal places. (a) $2x^2-2x-3=0$ (b) $x(3x-4)=2$.

The fact that the solution is wanted correct to 2 dp is an indication that the formula should be used.

(a) $2x^2-2x-3=0$

Remembering that the formula applies to $ax^2+bx+c=0$ we see that $a=2$, $b=-2$, $c=-3$.

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}=\frac{2\pm\sqrt{(4-(-24))}}{4}$$

$$=\frac{2+\sqrt{28}}{4} \text{ or } \frac{2-\sqrt{28}}{4}$$

From tables $\sqrt{28}=5.292$

$$x=\frac{7.292}{4}=1.823$$

$$\text{or } x=\frac{-3.292}{4}=-0.823$$

The solutions are 1.82 and -0.82.

(b) $x(3x-4)=2$

Written in the standard form as $3x^2-4x-2=0$, $a=3$, $b=-4$, $c=-2$.

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

$$=\frac{4\pm\sqrt{(16-(-24))}}{6}$$

$$=\frac{4\pm\sqrt{40}}{6} \qquad (\sqrt{40}=6.325)$$

$$x=\frac{4+6.325}{6}=1.721$$

$$\text{or } x=\frac{4-6.325}{6}=-0.388$$

The roots are 1.72 and -0.39.

1.6 The Graphical Solution of Quadratic Equations

Points on the locus of the equation $y=f(x)$ can be represented by ordered pairs (x, y). When $f(x)$ is a quadratic function the locus is a characteristic curve called a parabola.

When the graph of a quadratic function $y=f(x)$ is drawn, the solution of the equation $f(x)=0$ can be read directly from the graph by finding the two values of x at which the curve crosses the x-axis. From the same graph the solutions of other equations, such as $f(x)=2$, and inequalities such as $f(x)\leqslant 5$ can be obtained.

Example 1.13

Draw the graph of $y=(2x+3)(x-2)$ for values of x in the domain $\{-3\leqslant x\leqslant 3\}$. From your graph find (a) the values of x for which $(2x+3)(x-2)\geqslant 0$, (b) the solution of $2x^2-x=11$.

A table of values of x and y is constructed for the given domain.

x	-3	-2	-1	0	1	2	3
$2x+3$	-3	-1	1	3	5	7	9
$x-2$	-5	-4	-3	-2	-1	0	1
y	15	4	-3	-6	-5	0	9

The y-axis may be drawn in the middle of the page, but more space is needed above the x-axis than below it since the values of y are in the range $\{-6\leqslant y\leqslant 15\}$. In an examination if the scales are specified in a question it is important to obey the instructions exactly.

Points representing the ordered pairs (x, y) are plotted and joined by as smooth a curve as possible using a hard pencil with a sharp point.

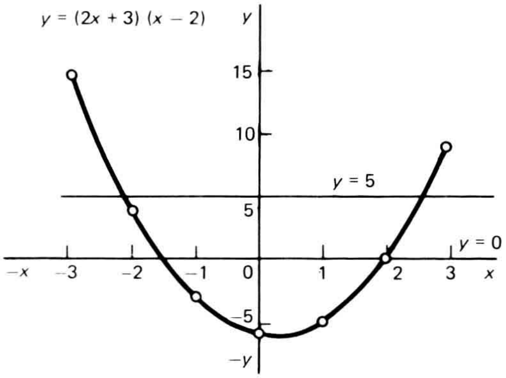

Figure 1.1

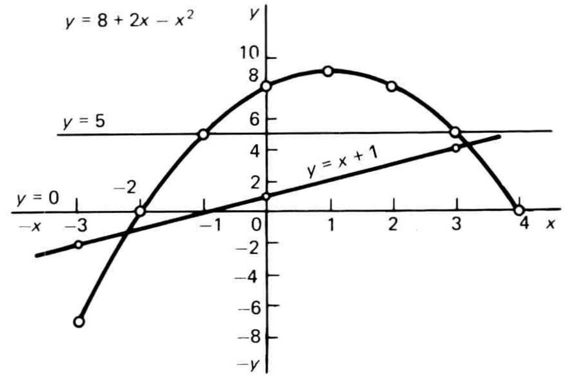

Figure 1.2

(a) The curve crosses the line $y = 0$ at $x = 2$ and $x = -1.5$. For negative values of y (below the x-axis) the curve is between these two values, and for positive values of y the curve lies to the right of $x = 2$ and to the left of $x = -1.5$.

The solution of $(2x+3)(x-2) \geqslant 0$ is $x \geqslant 2$, $x \leqslant -1.5$.

(b) Multiplying out

$$(2x+3)(x-2) = 2x^2 + 3x - 4x - 6$$
$$= 2x^2 - x - 6$$
$$2x^2 - x = 11 \Rightarrow 2x^2 - x - 6 = 5$$

The curve represents $y = 2x^2 - x - 6$, and the values of x satisfying $2x^2 - x = 11$ are the values of x satisfying $y = 5$.

The line $y = 5$ is drawn and the values of x at which the curve cuts this line are found to be 2.6 and -2.15.

The solution set of $2x^2 - x = 11$ is $\{2.6, -2.15\}$.

Example 1.14

Draw the graph of the function $f(x) = 8 + 2x - x^2$, for the domain $\{x: -3 \leqslant x \leqslant 4\}$, and use the graph to solve (a) $8 + 2x - x^2 \geqslant 0$, (b) $3 + 2 - x^2 \leqslant 0$, (c) $7 + x - x^2 = 0$.

Let $y = f(x)$.

Make a table of values of x and y for the domain $-3 \leqslant x \leqslant 4$.

x	-3	-2	-1	0	1	2	3	4
$2x$	-6	-4	-2	0	2	4	6	8
$-x^2$	-9	-4	-1	0	-1	-4	-9	-16
$+8$	8	8	8	8	8	8	8	8
y	-7	0	5	8	9	8	5	0

(a) The graph (figure 1.2) is different from the previous example, which had the vertex of the parabola at the lowest point. This time the vertex is at the top, because the coefficient of x^2 is negative. The curve crosses the x-axis at $(-2, 0)$ and $(4, 0)$, and so $8 + 2x - x^2 = 0$ when $x = -2$ and 4.

For all points on the curve above the axis, x has values between -2 and 4.

The solution of $8 + 2x - x^2 \geqslant 0$ is $\{x: -2 \leqslant x \leqslant 4\}$.

(b) $3 + 2x - x^2 = 8 + 2x - x^2 - 5$

The solution of $3 + 2x - x^2 \leqslant 0$ is the solution of

$$8 + 2x - x^2 - 5 \leqslant 0$$
$$\text{or } y \leqslant 5$$

The line $y = 5$ is drawn on the graph, and the values of x where the curve meets this line are found to be -1 and 3. For values of $y < 5$, the curve is to the left of $x = -1$ and to the right of $x = 3$, and so the solution of the inequality $3 + 2x - x^2 \leqslant 0$ is $\{x: x \leqslant -1, x \geqslant 3\}$.

(c) $7 + x - x^2 = 0$

We have drawn the graph of $y = 8 + 2x - x^2$, and in order to solve a quite different equation we must find the relation between the two functions.

$$7 + x - x^2 = (8 + 2x - x^2) - (1 + x)$$
$$\therefore\ 7 + x - x^2 = 0 \Rightarrow 8 + 2x - x^2 = 1 + x$$

The solutions are the points on the graph of $y = 8 + 2x - x^2$ for which $y = 1 + x$.

The straight line $y = 1 + x$ is drawn on the same axes. From the graph, the line and the curve intersect at $x = -2.2$ and 3.2. The solution set for $7 + x - x^2 = 0$ is $\{-2.2, 3.2\}$.

Exercise 1.2

A

(1) State the solution sets
(a) $(x-3)(x+2)=0$, (b) $(x-6)(x-4)<0$,
(c) $(x+3)(x+1)>0$, (d) $(2y-3)(3y+2)=0$,
(e) $(4x-1)(3x+6)=0$, (f) $z(z-1)\geqslant 0$,
(g) $2t(3t-2)\leqslant 0$, (h) $x^2<16$, (i) $y^2\geqslant 25$.
(2) Solve the equations
(a) $y^2-1=0$, (b) $2x^2-8=0$, (c) $x^2=9x$,
(d) $t^2-t-12=0$, (e) $y^2+7y+10=0$,
(f) $3x^2-14x=5$, (g) $6y^2-17y+12=0$,
(h) $(x-3)^2=16$.
(3) Which of the following statements are necessarily true, when x and y are real numbers?
(a) $(x+5)(x-3)>0 \Rightarrow x>3$, (b) $x^2>x \Rightarrow x>1$,
(c) $x>4 \Rightarrow (x-4)(x+2)>0$,
(d) $x^2-3x<0 \Rightarrow 0<x<3$,
(e) $\dfrac{x}{y}<3 \Rightarrow (x-3y)^2>0$, (f) $x>y \Rightarrow x^2>y^2$,
(4) Prove that $\dfrac{n}{n+1}<\dfrac{n+1}{n+2}$ $(n\in N)$

B

(1) (a) Find the negative number x such that $3x^2=x+2$. (b) Given that t is a real number, factorise t^3-t and hence find the solution set for $t^3-t=0$.
(2) Solve, correct to 2dp, the following equations
(a) $2x^2-x-5=0$, (b) $5x^2+7x+2=0$,
(c) $3x^2=4x+7$.
(3) Find the solutions correct to 2dp of
(a) $x(2x+3)=2$, (b) $\dfrac{2-3x}{x}=3x$, (c) $(x-3)^2=8$.
(4) Graphically, or otherwise, find the minimum value of the following functions
(a) $3x^2+5x-6$, (b) x^2-4.
(5) Find the greatest value of (a) $4+2x-3x^2$, (b) $1-9x-3x^2$.
(6) (i) In each case, write down a quadratic equation with the given roots and expand it as a trinomial
(a) $(1,-3)$, (b) $(-2,-1)$, (c) $(2,\frac{1}{2})$, (d) $(-\frac{1}{2},\frac{1}{3})$.
(ii) If $2x^2-ax+b=0$ is satisfied by $x=-\frac{1}{2}$ and $x=3$, find the integers a and b.
(7) Draw the graph of the function
$f(x)=2x^2-3x-2$ for $-2\leqslant x\leqslant 3$
and use your graph to solve
(a) $2x^2-3x-2=0$, (b) $2x^2-3x-2<0$,
(c) $2x^2-3x+1=0$, (d) $2x^2-4x+2=0$.
(8) A projectile is h metres above the point of projection after t seconds, where $h=30t-4t^2$. Draw a graph of $t\to h$ for $\{t:0\leqslant t\leqslant 8\}$. (a) What is the greatest height attained by the projectile? (b) Find the period of time for which the height is more than 20 m.
(9) Given $f(x)=6+5x-3x^2$, plot a graph of the function for the domain $\{x:-1\leqslant x\leqslant 3\}$ and estimate (a) the greatest positive value of the function, (b) the roots of the equation $5+2x-3x^2=0$, (c) the range of values of x for which $2-3x^2$ has a positive value for the given domain.
(10) On the same Cartesian axes draw graphs of the ordered pairs
$A=\{(x,y):y=x^2,\ -2\leqslant x\leqslant 5\}$,
$B=\{(x,y):y=3x+4,\ -2\leqslant x\leqslant 5\}$.
(a) Use your graph to find members of the set $A\cap B$.
(b) Shade in the region represented by $x\leqslant 0$, $y\geqslant x^2$, $y\leqslant 3x+4$.

1.7 Harder Equations

Simultaneous Linear and Quadratic Equations

Example 1.15

Solve the simultaneous equations
$2x^2-x=y+6$
$3x+y-6=0$
Algebraic method. The calculation is in four stages.
(i) Express y in terms of x from the linear equation
$y=6-3x$
(ii) Substitute this value of y in the first equation to obtain a quadratic equation in x
$2x^2-x=6-3x+6$
(iii) Rearranging the terms gives
$2x^2+2x-12=0$
$x^2+x-6=0$
$(x+3)(x-2)=0$
which leads to the solution $x=2$ or $x=-3$.
(iv) Substitute these values of x in $3x+y-6=0$ to find the corresponding values of y.
When $x=2$, $6+y-6=0 \Rightarrow y=0$
When $x=-3$, $-9+y-6=0 \Rightarrow y=15$
The simultaneous equations are satisfied by ordered pairs $(2, 0)$ and $(-3, 15)$.
Note: Simultaneous equations may be solved graphically as in examples 1.13 and 1.14.

Example 1.16

Find the value of mn when $m^2+n^2=19$ and $m+n=7$.
Since the question requires the product mn and not

separate values the identity
$(m+n)^2 \equiv m^2+n^2+2mn$ is used.

$7^2 = 19+2mn$

$49-19 = 2mn \Rightarrow mn = \underline{15}$

Example 1.17

What is the value of $\frac{1}{u}+\frac{1}{v}$ when $u+v = 10$ and $uv = 6$?

$$\frac{1}{u}+\frac{1}{v} = \frac{v+u}{uv} = \frac{10}{6} = \underline{1\tfrac{2}{3}}.$$

Example 1.18

Find the values of x which satisfy the equation

$y = \frac{1+3x}{1-5x}$ when $y = -x$

When $y = -x$, $-x = \frac{1+3x}{1-5x}$

Multiplying the equation by $1-5x$ leads to the quadratic equation

$-x(1-5x) = 1+3x$

$-x+5x^2 = 1+3x \Rightarrow 5x^2-4x-1 = 0.$

This factorises as $(5x+1)(x-1) = 0$ leading to the solution $\underline{x = -\frac{1}{5}, x = 1}$.

Fractional Equations

Example 1.19

Solve the equations

(a) $\frac{1}{x}-\frac{1}{x+1} = \frac{1}{x+2}$

(b) $\frac{1}{x-3}+\frac{2}{x+2} = \frac{2}{5}$

(a) Each term must be multiplied by the product $x(x+1)(x+2)$

$$\frac{x(x+1)(x+2)}{x}-\frac{x(x+1)(x+2)}{x+1} = \frac{x(x+1)(x+2)}{x+2}$$

Cancelling leaves the equation as

$(x+1)(x+2)-x(x+2) = x(x+1)$

or $x^2+2x+x+2-x^2-2x = x^2+x$

and bringing the terms to the left-hand side yields the quadratic equation

$-x^2+2 = 0 \Rightarrow x^2 = 2 \Rightarrow \underline{x = \pm\sqrt{2}}$

(b) $\frac{1}{x-3}+\frac{2}{x+2} = \frac{2}{5}$

Multiply each term by $5(x-3)(x+2)$

$$\frac{5(x-3)(x+2)}{x-3}+\frac{2\times 5(x-3)(x+2)}{x+2} = \frac{2\times 5(x-3)(x+2)}{5}$$

$5(x+2)+10(x-3) = 2(x-3)(x+2)$

$5x+10+10x-30 = 2(x^2-3x+2x-6)$

$15x-20 = 2x^2-2x-12$

which leads to the quadratic equation

$2x^2-17x+8 = 0 \Rightarrow (2x-1)(x-8) = 0$

$2x-1 = 0 \Rightarrow x = \frac{1}{2}$, $x-8 = 0 \Rightarrow x = 8$

giving the solution $\underline{\{\frac{1}{2}, 8\}}$.

Example 1.20

For what values of s/t is the expression $3s^2-11st+6t^2$ equal to 0?

Dividing each term by t^2 will give a quadratic equation with s/t as the variable

$$3\left(\frac{s}{t}\right)^2-11\left(\frac{s}{t}\right)+6 = 0$$

Let $\frac{s}{t} = x$, then

$3x^2-11x+6 = 0$

$(3x-2)(x-3) = 0 \Rightarrow x = \frac{2}{3}$ or $x = 3$

$3s^2-11st+6t^2 = 0 \Rightarrow \underline{\frac{s}{t} = \frac{2}{3} \text{ or } 3}$

1.8 Conditional Equations and Identities

The equation $(a+b)^2 = a^2+2ab+b^2$ is true for all values of a and b, and is an identity.

However, the equation $(a+b)^2 = a^2$ is only true for all values of a provided $b = 0$, and it is called a conditional equation.

Example 1.21

State whether the following equations are satisfied by one or two values of x or neither of these. Give the solution set in each case.

(1) $(2x-3)^2 = 4x^2-12x+9$
(2) $(2x-3)^2 = 2x^2+12x+9$
(3) $(2x-3)^2 = 4x^2-12x-9$
(4) $(2x-3)^2 = 4x^2-9$
(5) $(2x-3)^2 = 25$

$(2x-3)(2x-3) \equiv 4x^2-12x+9$

(1) Equation 1 is an identity and true for all values of x. The solution set is $\underline{\{x : x \in R\}}$.