Carla M. A. Pinto
J. A. Tenreiro Machado

# Linear Algebra

## — Selected Problems

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# Contents

# Chapter 1
# Vector spaces

## 1.1 Fundamentals

**Definition 1.1.** A vector space $\mathscr{E}$ over a field $\mathscr{K}$ is a set $\mathscr{E}$ on which the operations **addition** $\oplus : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and **scalar multiplication** $\otimes : \mathscr{K} \times \mathscr{E} \to \mathscr{E}$ satisfy, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{E}$, and $\alpha, \beta, 1 \in \mathscr{K}$:

| | | |
|---|---|---|
| (A1) | Addition is commutative | $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ |
| (A2) | Addition is associative | $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ |
| (A3) | Additive identity $\mathbf{0}$ exists | $\mathbf{v} \oplus 0 = \mathbf{v}$ |
| (A4) | Additive inverse $-\mathbf{v}$ exists | $\mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$ |
| (M1) | Multiplication is assocative | $(\alpha\beta) \otimes \mathbf{v} = \alpha(\beta\mathbf{v})$ |
| (M2) | Multiplicative identity exists | $1 \otimes \mathbf{v} = \mathbf{v}$ |
| (D1) | Distributive law for scalars | $(\alpha + \beta) \otimes \mathbf{v} = \alpha\mathbf{v} \oplus \beta\mathbf{v}$ |
| (D2) | Distributive law for vectors | $\alpha(\mathbf{v} \oplus \mathbf{w}) = \alpha\mathbf{v} \oplus \beta\mathbf{w}$ |

**Table 1.1** Properties of a vector space $\mathscr{E}$.

*Remark 1.1.* By abuse of notation and when convenient, it is written $+$ instead of $\oplus$ and $\times$ instead of $\otimes$.

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**Definition 1.2.** A nonempty subset $\mathscr{F} \subseteq \mathscr{E}$ is a subspace if $\mathscr{F}$ is a vector space using the operations of addition and multiplication defined on $\mathscr{E}$.

**Definition 1.3.** A nonempty subset $\mathscr{F} \subseteq \mathscr{E}$ is a subspace of $\mathscr{E}$ if and only if:

1. $\mathscr{F} \neq \emptyset$.
2. $\forall \mathbf{u}, \mathbf{v} \in \mathscr{F},\ \forall \alpha, \beta \in \mathscr{K} :\ \alpha\mathbf{u} + \beta\mathbf{v} \in \mathscr{F}$.

**Definition 1.4.** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a set of vectors of a vector space $\mathscr{E}$. A vector $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if

$$\mathbf{v} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \ldots + \alpha_k \mathbf{v_k}$$

for some scalars $\alpha_1, \ldots, \alpha_k$.

**Definition 1.5.** The set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a vector space $\mathscr{E}$ is the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, denoted by $\langle \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle$.

**Definition 1.6.** Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in the vector space $\mathscr{E}$. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the remaining $k-1$ vectors.

**Lemma 1.1.** *The set of vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *is linearly independent if and only if whenever*

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

*it follows that* $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$.

**Theorem 1.1.** *Let* $\mathbf{b} = (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k)$ *be a set of vectors in a vector space* $\mathscr{F}$. *The subset* $\mathbf{b}$ *is a basis of* $\mathscr{F}$ *if and only if the set* $\mathbf{b}$ *is linearly independent and spans* $\mathscr{F}$.

**Definition 1.7.** The dimension of a vector space $\mathscr{E}$ is the cardinality of any basis of $\mathscr{E}$ and is denoted by $\dim \mathscr{E}$. $\mathscr{E}$ is finite-dimensional if it is the zero subspace $\{\mathbf{0}\}$ or if it has a basis of finite cardinality. Otherwise it is called infinite-dimensional.

**Theorem 1.2.** *Any two basis for a vector space* $\mathscr{E}$ *have the same cardinality.*

**Theorem 1.3.** *Let $\mathscr{E}$ be a vector space. If $\mathscr{C} = \{F_i : i \in \mathscr{K}\}$ is a collection of subspaces of $\mathscr{E}$, then*

$$\bigoplus_{i\in\mathscr{K}} F_i; \quad \bigcap_{i\in\mathscr{K}} F_i$$

*are subspaces.*

**Theorem 1.4.** *If $\mathscr{F}$ and $\mathscr{G}$ are subspaces of a vector space $\mathscr{E}$, then $\mathscr{F} \cup \mathscr{G}$ is a subspace if and only if $\mathscr{F} \subseteq \mathscr{G}$ or $\mathscr{G} \subseteq \mathscr{F}$.*

**Theorem 1.5.** *If $\mathscr{F}$ is a subspace of a vector space $\mathscr{E}$, then there exists a subspace $\mathscr{G}$ such that:*

$$\mathscr{E} = \mathscr{F} \oplus \mathscr{G}$$

**Theorem 1.6.** *Let $\mathscr{F}$ be a subspace of a finite-dimensional vector space $\mathscr{E}$.*

***a)*** *Suppose that $\mathscr{F}$ is a proper subspace ($\mathscr{F} \subset \mathscr{E}$), then $\dim \mathscr{F} < \dim \mathscr{E}$.*
***b)*** *Suppose that $\dim \mathscr{F} = \dim \mathscr{E}$, then $\mathscr{F} = \mathscr{E}$.*

**Theorem 1.7.** *If $\mathscr{F}$ and $\mathscr{G}$ are subspaces of a vector space $\mathscr{E}$, then:*

$$\dim(\mathscr{F} \oplus \mathscr{G}) = \dim \mathscr{F} + \dim \mathscr{G} - \dim \mathscr{F} \cap \mathscr{G}$$

*Moreover, if $\mathscr{F} \cap \mathscr{G} = \emptyset$ then $\dim \mathscr{F} \cap \mathscr{G} = 0$ and*

$$\dim \mathscr{F} \oplus \mathscr{G} = \dim \mathscr{F} + \dim \mathscr{G}$$

## 1.2 Worked Examples

**Problem 1.1** Let $\mathscr{E} = \mathbb{R}^2$ and $\mathscr{K} = \mathbb{R}$. Addition $\oplus$ is defined as:

$$\begin{aligned} \oplus : \quad & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \\ & (x_1, y_1) \oplus (x_2, y_2) \to (x_1 + 3x_2, y_1 - y_2) \end{aligned}$$

Is $\mathscr{E}$ a vector space over the field $\mathscr{K}$?

**Resolution**

For $\mathscr{E}$ to be a vector space over the field $\mathscr{K}$ it must satisfy all properties in Table 1.1.

Let's start with property (A1). Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$. Then:

$$\mathbf{u} \oplus \mathbf{v} = (x_1, y_1) \oplus (x_2, y_2) = (x_1 + 3x_2, y_1 - y_2)$$

On the other hand, it is obtained:

$$\mathbf{v} \oplus \mathbf{u} = (x_2, y_2) \oplus (x_1, y_1) = (x_2 + 3x_1, y_2 - y_1)$$

Thus $\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}$, and the property fails. One concludes that $\mathscr{E}$ is not a vector space over the field $\mathscr{K}$.

**Problem 1.2** Let $\mathscr{E} = \mathbb{R}^3$ and $\mathscr{A} = \{(1,0,1), (0,1,1), (0,-1,-5+a)\}$.

**a)** Show that for $a = 1$, $\mathscr{A}$ is a subset of linearly independent vectors.
**b)** Write $\mathbf{w} = (x, y, z)$ as a linear combination of the three vectors of $\mathscr{A}$.

**Resolution**

**a)** For $a = 1$, $\mathscr{A} = \{(1,0,1), (0,1,1), (0,-1,-4)\}$. To prove that these three vectors are linearly independent, one must show that whenever

$$\alpha(1,0,1) + \beta(0,1,1) + \gamma(0,-1,-4) = (0,0,0) \tag{1.1}$$

it follows that $\alpha = \beta = \gamma = 0$.
Expression (1.1) is equivalent to:

$$(\alpha, \beta - \gamma, \alpha + \beta - 4\gamma) = (0,0,0)$$

$$\alpha = 0; \ \beta = \gamma; \ \gamma = 0$$

Thus $\alpha = \beta = \gamma = 0$, so the vectors are linearly independent.
**b)** To write $\mathbf{w}$ as a linear combination of the three vectors of $\mathscr{A}$, one must solve:

$$\alpha(1,0,1) + \beta(0,1,1) + \gamma(0,-1,-4) = (x, y, z) \tag{1.2}$$

Expression (1.2) is equivalent to:

$$(\alpha, \beta - \gamma, \alpha + \beta - 4\gamma) = (x, y, z)$$

$$\alpha = x; \ \beta = y + \gamma; \ \gamma = -\tfrac{z-x-y}{3}$$

Thus

$$(x, y, z) = x(1,0,1) - \frac{z - x - 4y}{3}(0,1,1) - \frac{z - x - y}{3}(0,-1,-4)$$

**Problem 1.3** Are the following sets basis of the corresponding vector spaces? If yes, compute their dimension.

**a)** $\{(1,-1), (2,3)\}$ of $\mathscr{E} = \mathbb{R}^2$.
**b)** $\{(-1,-1,1), (2,1,0), (1,0,-1)\}$ of $\mathscr{E} = \mathbb{R}^3$.

**Resolution**

**a)** To prove that $\{(1,-1),(2,3)\}$ is a basis of $\mathscr{E}=\mathbb{R}^2$, one must check if the two vectors are linearly independent and if they span $\mathbb{R}^2$. As it is known that $\dim\mathbb{R}^2=2$, then it is sufficient to show that the two vectors are linearly independent, since as they are two and two is the cardinality of a basis of $\mathbb{R}^2$, then it is proved that they are elements of a basis of $\mathbb{R}^2$.
The two vectors are linearly independent if and only if

$$\alpha(1,-1)+\beta(2,3)=(0,0)\Rightarrow\alpha=\beta=0$$

Thus

$$\begin{aligned}&(\alpha,-\alpha)+(2\beta,3\beta)=(0,0)\\&\Leftrightarrow\alpha+2\beta=0\wedge-\alpha+3\beta=0\\&\Leftrightarrow\alpha=-2\beta\wedge\beta=0\Leftrightarrow\alpha=0,\beta=0\end{aligned}$$

One concludes that the two vectors form a basis of $\mathbb{R}^2$, since they are linearly independent.

**b)** To prove that $\{(-1,-1,1),(2,1,0),(1,0,-1)\}$ is a basis of $\mathscr{E}=\mathbb{R}^3$, one must check if the three vectors are linearly independent and if they span $\mathbb{R}^3$. Thus:

$$\begin{aligned}&\alpha(-1,-1,1)+\beta(2,1,0)+\gamma(1,0,1)=(0,0,0)\\&\Rightarrow-\alpha+2\beta+\gamma=0\wedge-\alpha+\beta=0\wedge\alpha+\gamma=0\\&\qquad 0=0\wedge\beta=\alpha\wedge\gamma=-\alpha\end{aligned}$$

As $\alpha\in\mathbb{R}$, then the three vectors are linearly dependent. In fact, one can easily show that $(1,0,1)=(-1,-1,1)+(2,1,0)$, so the third vector is a linear combination of the two first vectors.

**Problem 1.4** Find the subspace of $\mathbb{R}_2[x]$ spanned by the vectors $\{1,x,x^2+x\}$.

**Resolution** A general vector of $\mathbb{R}_2[x]$ is of the form $a+bx+cx^2$. This vector is spanned by $\{1,x,x^2+x\}$ if it is written as a linear combination of them. Thus:

$$\begin{aligned}&\alpha\times1+\beta x+\gamma(x^2+x)=a+bx+cx^2\\&\Leftrightarrow\alpha+(\beta+\gamma)x+\gamma x^2=a+bx+cx^2\end{aligned}$$

Two polynomials are identical if the coefficients of the similar monomials are equal. Then:

$$\alpha=a;\ \beta+\gamma=b;\ \gamma=c\Leftrightarrow\alpha=a;\ \beta=b-c;\ \gamma=c$$

The subset spanned by $\{1,x,x^2+x\}$ is $A=\{a+(b-c)x+cx^2:\ a,b,c\in\mathbb{R}\}$.

**Problem 1.5** Let $\mathscr{F}=\{(x,y,z,t)\in\mathbb{R}^4:\ y+z+t=0\}$ and $\mathscr{G}=\{(x,y,z,t)\in\mathbb{R}^4:\ x+y=0\wedge z=2t\}$ be two subsets of the vector space $\mathbb{R}^4$. Compute a basis and the dimension of $\mathscr{F}\cap\mathscr{G}$.

**Resolution** Vectors in the subset $\mathscr{F} \cap \mathscr{G}$ must satisfy the conditions $y+z+t=0$ and $x+y=0 \wedge z=2t$, thus

$$\mathscr{F} \cap \mathscr{G} = \{(x,y,z,t) \in \mathbb{R}^4 : \ y+z+t=0 \wedge x+y=0 \wedge z=2t\}$$
$$\mathscr{F} \cap \mathscr{G} = \{(x,y,z,t) \in \mathbb{R}^4 : y=-z-t=-2t-t=-3t \wedge x=-y=3t \wedge z=2t\}$$
$$\mathscr{F} \cap \mathscr{G} = \{(3t,-3t,2t,t),\ t \in \mathbb{R}\} = \langle (3,-3,2,1) \rangle$$

The dimension is $\dim \mathscr{F} \cap \mathscr{G} = 1$, since $\mathscr{F} \cap \mathscr{G}$ is spanned by only one vector.

## 1.3 Proposed Exercises

**Exercise** 1.1
Let $\mathscr{E}$ be a non empty set and $\mathscr{K}$ a field. In the itens below, justify if $\mathscr{E}$ is a vector space over the field $\mathscr{K}$.

a) $\mathscr{E} = \mathbb{R}[x]$, set of polynomials in the variable $x$ and $\mathscr{K} = \mathbb{R}$. Addition is defined in the following way: let $\mathbf{a} = a_0 + a_1 x + \ldots + a_m x^m$ and $\mathbf{b} = b_0 + b_1 x + \ldots + b_n x^n$ be two elements of $\mathbb{R}[x]$. It is assumed, without loss of generality that $m \leq n$, then:

$$\mathbf{a}+\mathbf{b} = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_m+b_m)x^m + \cdots + (a_n+b_n)x^n$$

where $a_{m+1} = \cdots = a_n = 0$.
The scalar multiplication $\mathscr{K} \times \mathscr{E} \to \mathscr{E}$ is given by:

$$\alpha \mathbf{a} = \alpha a_0 + (\alpha a_1)x + \cdots + (\alpha a_n)x^n$$

where $\alpha \in \mathbb{R}$ and $a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{R}[x]$.

b) $\mathscr{E} = \mathbb{Z}$ and $\mathscr{K} = \mathbb{R}$ with the usual addition of integers and the multiplication of a vector by a real number.

c) $\mathscr{E} = \mathscr{C}[a,b] = \{f : [a,b] \to \mathbb{R} : f \text{ is a continuous function}\}$ and $\mathscr{K} = \mathbb{R}$, with the usual addition of functions and the usual multiplication of a scalar by a function.

d) $\mathscr{E} = \mathbb{R}^+$ and $\mathscr{K} = \mathbb{R}$, with addition of two vectors and multiplication by a scalar defined by:

$$\oplus : \ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$
$$(x,y) \to x \oplus y = \frac{x}{y}$$

$$\otimes : \ \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$$
$$(\alpha, x) \to \alpha \otimes x = x^{\alpha}$$

e) Let $\mathscr{K}$ be a field and $\mathscr{E} = \mathscr{K}^n$ the set of the $n$ ordered -uples of elements of $\mathscr{K}$ given by:

$$\mathscr{K}^n = \{\mathbf{a} = (a_1, a_2, \ldots, a_n) : \ a_i \in \mathscr{K}\}$$

Addition is defined by:

$$\mathbf{a}+\mathbf{b}=((a_1+b_1),(a_2+b_2),\cdots,(a_n+b_n))$$

for all $(a_1,a_2,\ldots,a_n)$, $(b_1,b_2,\ldots,b_n)\in\mathscr{K}^n$.
Multiplication by a scalar $\alpha\in\mathscr{K}$ is defined by:

$$\alpha\mathbf{a}=(\alpha a_1,\alpha a_2,\cdots,\alpha a_n)$$

where $(a_1,a_2,\cdots,a_n)\in\mathscr{K}^n$.

f) $\mathscr{E}=\mathbb{R}^3$ and $\mathscr{K}=\mathbb{R}$, with the usual addition and the multiplication by a scalar defined as follows:

$$c(x_1,x_2,x_3)=(0,0,cx_3)$$

g) $\mathscr{E}=\mathbb{R}^2$ and $\mathscr{K}=\mathbb{R}$, with the usual multiplication by a scalar and addition defined by:

$$(x_1,x_2)\oplus(y_1,y_2)=(x_1+2y_1,x_2+y_2)$$

h) $\mathscr{E}=\mathbb{R}^2$ and $\mathscr{K}=\mathbb{R}$, with the usual addition and multiplication by a scalar.

i) $\mathscr{E}=\mathbb{R}^2$ and $\mathscr{K}=\mathbb{R}$, with addition and multiplication by a scalar given by:

$$\begin{aligned}\oplus:\ &\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+\\ &(x,y)\to x\oplus y=\tfrac{x}{y}\end{aligned}$$

$$\begin{aligned}\otimes:\ &\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}^+\\ &(\alpha,x)\to\alpha\otimes x=x^\alpha\end{aligned}$$

**Exercise** 1.2
Let $\mathscr{E}$ be a vector space, $\mathbf{x}=(x_1,\cdots,x_n)$ and $\mathbf{y}=(y_1,\cdots,y_n)$ be two vectors of $\mathscr{E}$, where $x_i,y_i>0$, $i=1,\cdots,n$, and $\lambda\in\mathbb{R}$. Addition and multiplication by a scalar are defined, respectively, as follows:

$$\begin{aligned}\mathbf{x}\oplus\mathbf{y}&=(x_1y_1,\cdots,x_ny_n)\\ \lambda\otimes\mathbf{x}&=(x_1^\lambda,\cdots,x_n^\lambda)\end{aligned}$$

a) Let $\mathbf{0}$ be the additive identity in $\mathscr{E}$. Then:

**A)** $\mathbf{0}=(1,1,\cdots,1)$.
**B)** $\mathbf{0}=(0,0,\cdots,0)$.
**C)** $\mathbf{0}=(x_1^{-1},x_2^{-1},\cdots,x_n^{-1})$.
**D)** Other.

b) Let 1 be the multiplicative identity in $\mathbb{R}$. Then:

**A)** $1=1$.
**B)** $1=(1,1,\ldots,1)$.
**C)** $1=0$.
**D)** Other.

c) Let $-\mathbf{x}$ be the additive inverse in $\mathscr{E}$ of $\mathbf{x}$. Then:

**A)** $-\mathbf{x} = (\ln x_1, \ln x_2, \ldots, \ln x_n)$.
**B)** $-\mathbf{x} = (-x_1, -x_2, \ldots, -x_n)$.
**C)** $-\mathbf{x} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$.
**D)** Other.

**Exercise** 1.3
Let $\mathscr{E}$ be a real vector space. Are the following vectors independent in the corresponding vector space? Justify.

a) $(3,1)$, $(4,-2)$ and $(7,2)$ in $\mathscr{E} = \mathbb{R}^2$.
b) $(0,-3,1)$, $(2,4,1)$ and $(-2,8,5)$ in $\mathscr{E} = \mathbb{R}^3$.
c) $(-1,2,0,2)$, $(5,0,1,1,)$ and $(8,-6,1,-5)$ in $\mathscr{E} = \mathbb{R}^4$.
d) $\mathbf{u} = 1$, $\mathbf{v} = 1-x$ and $\mathbf{w} = (1-x)^2$ in $\mathscr{E} = \mathbb{R}_2[x]$ (set of polynomials of degree less or equal to 2).
e) $\mathbf{u}(x) = e^x$ and $\mathbf{v}(x) = e^{2x}$, for all $x \in \mathbb{R}$, in $\mathscr{E} = \mathbb{F}(\mathbb{R})$, where $\mathbb{F}(\mathbb{R})$ is the set of applications of $\mathbb{R}$ in $\mathbb{R}$.

**Exercise** 1.4
Let $\mathbf{a} = (1,1,1,0)$, $\mathbf{b} = (0,1,1,1)$, $\mathbf{c} = (1,1,0,0)$, $\mathbf{d} = (x,y,z,t)$ be vectors in $\mathbb{R}^4$. These vectors are linearly independent if and only if:

**A)** $x-y+t \neq 0$.
**B)** $x+y-z \neq 0$.
**C)** $x+z-t \neq 0$.
**D)** None of the above.

**Exercise** 1.5
Let $(1,0,-1)$, $(1,1,0)$, $(k,1,-1)$ be three vectors of $\mathbb{R}^3$. For what values of $k \in \mathbb{R}$ are these vectors linearly independent?

**A)** $k \neq -2$.
**B)** $k \neq 2$.
**C)** $k \neq -1$.
**D)** $k \neq 1$.

**Exercise** 1.6
Consider the following polynomials of the vector space $\mathbb{R}_3[x]$:

$$u(x) = x^3 + 4x^2 - 2x + 3, \quad v(x) = x^3 + 6x^2 - x + 4, \quad w(x) = 3x^3 + 8x^2 - ax + b$$

where $a,\ b \in \mathbb{R}$. These polynomials are linearly dependent if:

**A)** $a \neq 8 \wedge b \neq 7$.
**B)** $a \neq 8 \wedge b \in \mathbb{R}$.
**C)** $a \in \mathbb{R} \wedge b \neq 7$.
**D)** None of the above.

**Exercise** 1.7
Let $\mathscr{E}$ be the vector space of all real-valued functions of one real variable. Consider the functions $f_1,\ f_2,\ f_3,\ g_1,\ g_2,\ g_3 \in \mathscr{E}$, such that $f_1(t) = e^{2t}$, $f_2(t) = t^2$, $f_3(t) = t$, $g_1(t) = \sin t$, $g_2(t) = \cos t$, $g_3(t) = t$. Then:

**A)** $f_1,\ f_2,\ f_3$ are linearly dependent.
**B)** $g_1,\ g_2,\ g_3$ are linearly independent.
**C)** $g_1,\ g_2,\ g_3$ are linearly dependent.
**D)** $f_2,\ f_3$ are linearly dependent.

**Exercise** 1.8
Let $\mathscr{E}$ be the vector space of all real-valued functions of one real variable. Consider the functions $f_1,\ f_2,\ f_3 \in \mathscr{E}$, such that $f_1(x) = \sin x + 3\sin 2x$, $f_2(x) = 2\sin x + \sin 3x$, $f_3(x) = 2\sin 2x$. Then:

**A)** $f_1,\ f_2,\ f_3$ are linearly dependent.
**B)** $f_1,\ f_2,\ f_3$ are linearly independent.

**Exercise** 1.9
Let $\mathbf{x} = (1,0,0)$, $\mathbf{y} = (0,1,0)$ and $\mathbf{z} = (0,0,1)$ be vectors of the real vector space $\mathbb{R}^3$. Let $\mathbf{w} = (a,b,c) \in \mathbb{R}^3$ be an arbitrary vector. Write $\mathbf{w}$ as a linear combination of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

**Exercise** 1.10
Let $\mathbf{x} = (1,a,2)$, $\mathbf{y} = (b,b^2,6)$ and $\mathbf{z} = (1,-a,2)$ be vectors of the real vector space $\mathbb{R}^3$.

a) Find the values of $a,\ b \in \mathbb{R}$ such that $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are linearly independent.
b) Let $b = 3$ and compute the values of $a \in \mathbb{R}$ such that $\mathbf{y} \in \langle \mathbf{x}, \mathbf{z} \rangle$.

**Exercise** 1.11
Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors of a real vector space $\mathscr{E}$. Show that the vectors $\alpha\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v} + \mathbf{w}$ are also linearly independent vectors.

**Exercise** 1.12
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors linearly independent vectors of a real vector space $\mathscr{E}$. Determine $\alpha \in \mathbb{R}$ such that the vectors $\alpha\mathbf{u} + 2\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly dependent.

**Exercise** 1.13
Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be linearly independent vectors of a real vector space $\mathscr{E}$. Compute $\alpha,\ \beta \in \mathbb{R}$ such that the vectors $\alpha\mathbf{u} + 2\mathbf{v} + 2\mathbf{w}$ and $\mathbf{u} + \beta\mathbf{v} - \mathbf{w}$ are linearly dependent.

**Exercise** 1.14
Consider the following real vector spaces and corresponding vectors. Show, for each case, if these vectors span the given vector spaces.

a) $A = \{(1,2),\ (0,-1),\ (1,-2)\}$, $\mathscr{E} = \mathbb{R}^2$.
b) $B = \{(1,0), (3,0)\}$, $\mathscr{E} = \mathbb{R}^2$.
c) $C = \{1,\ 2x,\ x^2+1,\ x^3-x\}$, $\mathscr{E} = \mathbb{R}_3[x]$.
d) $D = \{(1,3,0),\ (0,1,1),\ (1,1,1)\}$, $\mathscr{E} = \mathbb{R}^3$.

**Exercise** 1.15
Consider the vector $(\alpha, \beta, 1)$, where $\alpha, \beta \in \mathbb{R}$. This vector belongs to the space spanned by the vectors $(1,2,-1)$ and $(2,-1,2)$ for:

**A)** $\alpha = 1 \wedge \beta = 1$.
**B)** $\alpha = 2 \wedge \beta = 1$.
**C)** $\alpha = 3 \wedge \beta = 1$.
**D)** None of the above.

**Exercise** 1.16
Are the following sentences true or false? Justify.

a) Let $\mathscr{E}$ be a vector space of finite dimension $n$. Then any set of $n+1$ vectors is linearly dependent.
b) Let $\mathscr{E}$ be a vector space of finite dimension $n$. Then any set of vectors linearly independent is part of a basis.
c) A necessary condition for a family of vectors $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ of a vector space $\mathscr{E}$ to be independent is that none of the vectors is a multiple of any other.
d) The previous condition is sufficient.
e) A necessary condition for a family of vectors $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ of a vector space $\mathscr{E}$ to generate that space is that any family $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n, \mathbf{v})$ is dependent.
f) The previous condition is sufficient.
g) A necessary condition for a family of generators $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ of a vector space $\mathscr{E}$ to be dependent is that any smaller family of $n-1$ vectors generates $\mathscr{E}$.
h) The previous condition is sufficient.

**Exercise** 1.17
Consider the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3 \in \mathbb{R}^4$ such that $\mathbf{u}_1 = (1,0,-1,2)$, $\mathbf{u}_2 = (1,2,-5,0)$, $\mathbf{u}_3 = (1,1,-3,1)$. Choose the correct option.

a) The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are:

   **A)** linearly dependent.
   **B)** linearly independent.

b) The dimension of the vector subspace generated by the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3 \in \mathbb{R}^4$ is:

   **A)** 1.
   **B)** 2.
   **C)** 3.
   **D)** None of the above.

**Exercise** 1.18
Consider the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{u} = (1,3,-1)$, $\mathbf{v} = (2,1,3)$, $\mathbf{w} = (3,4,2)$. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are:

**A)** linearly dependent.
**B)** linearly independent.

**Exercise** 1.19
Consider the vector space $\mathscr{E}$ spanned by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ -2 \end{bmatrix} \right\}$$

The dimension of $\mathscr{E}$, $\dim \mathscr{E}$, is:

**A)** $\dim \mathscr{E} = 1$.
**B)** $\dim \mathscr{E} = 2$.
**C)** $\dim \mathscr{E} = 3$.
**D)** None of the above.

**Exercise** 1.20
Do the following vectors form a basis of the corresponding vector spaces? Justify.

a) $\{(1,2),\ (2,4)\}$ of $\mathscr{E} = \mathbb{R}^2$.
b) $\{(1,1,1),\ (1,0,3),\ (0,0,1)\}$ of $\mathscr{E} = \mathbb{R}^3$.
c) $\{(0,1,1,0),\ (1,-1,1,-1),\ (1,0,2,-1),\ (0,0,0,1)\}$ of $\mathscr{E} = \mathbb{R}^4$.
d) $\{1,\ 1+x,\ x^2+x^3\}$ of $\mathscr{E} = \mathbb{R}_3[x]$.
e) $\{1,\ x,\ x^2+x\}$ of $\mathscr{E} = \mathbb{R}_2[x]$.

**Exercise** 1.21
Let $\mathbb{R}_1[x]$ be the vector space of the real polynomials in $x \in \mathbb{R}$ with degree less or equal than one.

a) Prove that $\mathbf{b} = (1,x)$ and $\mathbf{b}' = (5x, 3+4x)$ are both basis of $\mathbb{R}_1[x]$.
b) The coordinates of the vectors $\mathbf{u} = (2,3)$ and $\mathbf{v} = (4,1)$ in $\mathbf{b}'$ are given by, respectively:

   **A)** $(4/15, 8/3)$, $(-4/15, 8/3)$.
   **B)** $(1/15, 2/3)$, $(-13/15, 4/3)$.
   **C)** $(-4/3, 13/15)$, $(8/3, -1/15)$.
   **D)** None of the above.

**Exercise** 1.22
The coordinates in the canonical basis of $\mathbb{R}^3$ of the vector $\mathbf{v}$ are $(4,-3,2)$. In the basis $\mathbf{b} = ((1,0,0),(1,1,0),(1,1,1))$ the same vector is written as:

**A)** $(5,-5,2)$.
**B)** $(-7,5,3)$.
**C)** $(4,-3,7)$.
**D)** $(7,-5,2)$.

**Exercise** 1.23
For what values of $k$ does the set of vectors $\{(1,k),\ (k,4)\}$ form a basis of $\mathbb{R}^2$?

**A)** $k = 2$.

**B)** $k \neq \pm 2$.
**C)** $k \in \mathbb{R}$.
**D)** $k = -2$.

**Exercise** 1.24
Consider the vectors $(1,1,-1)$, $(2,1,0)$, $(2,3,-1)$ of $\mathbb{R}^3$.

**a)** Are these vectors a set of generators of $\mathbb{R}^3$? If yes, write $(x,y,z) \in \mathbb{R}^3$ as a linear combination of these vectors.
**b)** Do the three vectors form a basis of $\mathbb{R}^3$? Why?

**Exercise** 1.25
Let $\mathscr{S} = \langle (2,1,1),(1,2,5),(1,-1,4) \rangle$. Find the dimension of $\mathscr{S}$ and find a basis for $\mathscr{S}$.

**Exercise** 1.26
Show that $\mathscr{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where

$$\mathbf{u}_1 = (1,0,1),\ \mathbf{u}_2 = (1,-1,2),\ \mathbf{u}_3 = (1,1,3))$$

is a basis of $\mathbb{R}^3$.

**Exercise** 1.27
Consider the vectores $\mathbf{u}_1 = (1,1,a)$, $\mathbf{u}_2 = (0,1,1)$, $\mathbf{u}_3 = (1,0,b)$ of $\mathbb{R}^3$.

a) Determine $a$ and $b$ such that $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ forms a basis of $\mathbb{R}^3$.
b) Consider $a = 0$ and $b = 1$. Express the vector $(1,2,0)$ in the basis $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

**Exercise** 1.28
Let $(1,\ 1+x^2,\ b(x))$ be a basis of $\mathbb{R}_2[x]$.

a) Compute $b(x)$.
b) Write the coordinates of $2x^2 - 7x$ in the basis $(1,\ 1+x^2,\ b(x))$.

**Exercise** 1.29
Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a basis of the vector space $\mathscr{E}$. Let $\mathbf{f}_1 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathbf{f}_2 = -\mathbf{e}_1 + \mathbf{e}_3$ and $\mathbf{f}_3 = \mathbf{e}_2$ be vectors of $\mathscr{E}$.

a) Show that $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is a basis of $\mathscr{E}$.
b) Express vector $2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$ in the basis $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$.
c) Determine a basis of $\mathscr{E}$ that includes the vectors $\mathbf{e}_1$ and $\mathbf{f}_1$.

**Exercise** 1.30
Verify which of these subsets are subspaces of the corresponding vector spaces.

a) $\{(x,y) \in \mathbb{R}^2 :\ x = 2y\}$ of $\mathscr{E} = \mathbb{R}^2$.
b) $\{(x,y) \in \mathbb{R}^2 :\ x = 2y+1\}$ of $\mathscr{E} = \mathbb{R}^2$.
c) $\{f$ real function of a real variable : $f(x).f'(x) = 1,\ \forall x \in \mathbb{R}\}$ of $\mathscr{E} = \{f :\ f$ is a real function of a real variable$\}$.
d) $\{f$ real function of a real variable : $f(x) = xf'(x),\ \forall x \in \mathbb{R}\}$ of $\mathscr{E} = \{f :\ f$is a real function of a real variable$\}$.