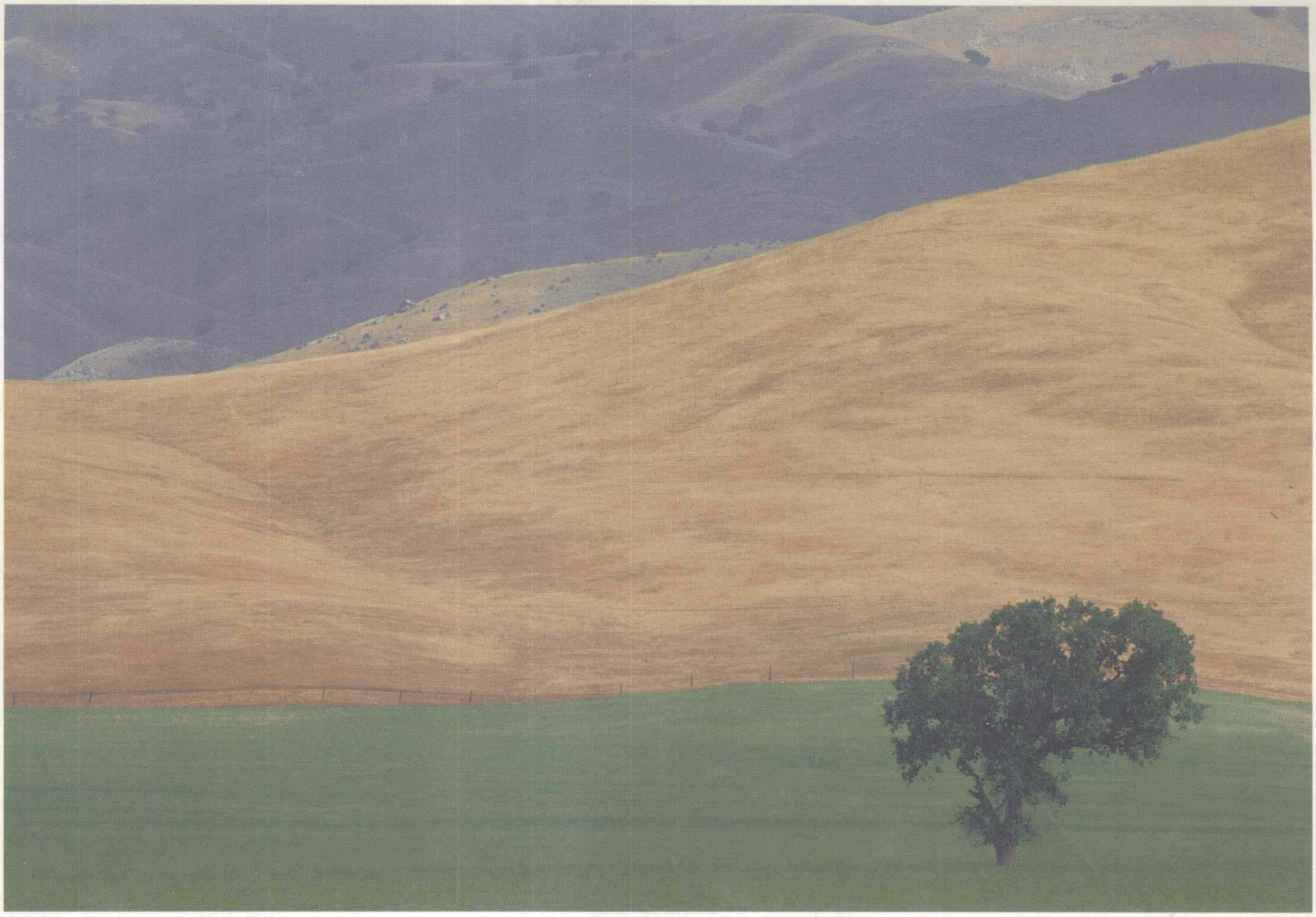

Mathematics

for the Liberal Arts Student

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MATHEMATICS

for the

Liberal Arts Student

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Preface

This text is for a one- or two-semester terminal course in mathematics. Such a course allows for leisurely exploration in place of drill—it is a course in mathematics appreciation. For this audience, one must constantly keep in mind the Hippocratic admonition, "First, do no harm." The authors believe that the spirit of mathematics can be communicated by means of simple ideas and problems without scaring or boring the students.

What are these ideas and problems? Counting patterns. Measuring likelihood. Interpreting polls. Interest payments. Population growth. Describing data. Different kinds of averages. Secret writing. Graphs (networks). Ruler-and-compass constructions. Incommensurable magnitudes. Playing with numbers. Logic puzzles. Infinity.

The history of the subject is integrated into the text. We maintain a historical consciousness throughout, but no attempt is made to offer a comprehensive history of mathematics or to include complete biographies of mathematicians. We introduce major mathematicians when their stories relate to the material at hand. Notes at the end of each chapter provide another opportunity to put a human face on mathematics.

Included as an appendix are short encounters, one-page treatments of various topics. Conversion between Celsius and Fahrenheit. Ladders for approximating $\sqrt{2}$, $\sqrt{3}$, etc. Collapsing compasses. The four-color problem.

We try to make the chapters as independent as possible, so that instructors can choose whatever pleases them. If the teacher doesn't like the material, the students certainly will not. We also strive to avoid superfluous generality, preferring illustrations to formulas. It is sometimes difficult for a mathematician to realize that an idea, depending on a parameter, may be understood in general by examining it for 5, yet be totally opaque when stated in terms of n. As John Stuart Mill said, "Not only may we reason from particulars to particulars without passing through generals, but we perpetually do so reason."

The typical college freshman will have all the prerequisites for the book. Nevertheless, an appendix is included for review of basic arithmetic concepts and notations for those students who are a bit rusty on these points.

The text meets the guidelines for two semesters of liberal arts mathematics established for the State of Florida's universities and community colleges. It has been used at Florida Atlantic University for both semesters for the past two years and at New Mexico State University for the first semester for the past year.

Some problems in the problem sets are in italics to indicate that they are difficult.

We would like to thank Lee Klingler and Heather Kanser, who have taught this material at Florida Atlantic University and helped us iron out some of the kinks.

If you have any questions or comments, e-mail them to `richman@fau.edu`.

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May 1999

Contents

Chapter 1

Counting

In this chapter, we will look at techniques for answering the question "how many?" without having to count each item, one by one. Numbers that are answers to the question "how many?" are called **cardinal numbers**. The study of the properties of cardinal numbers is one of the oldest and most fascinating branches of mathematics. Questions about cardinal numbers that have been worked on for thousands of years remain unanswered to this day.

1.1 One, two, three, ...

We begin by considering a topic very familiar to you: counting. You were taught to count at a very early age and could probably recite "one, two, three, four, five," even before you went to school. Counting the number of chairs in a room, the number of sheep in a field, and the number of students in a class are examples of the many occasions when one counts the number of objects in a set. The process usually begins, "one, two, three, ...," assigning a number to each object, and ends when every object has been assigned a number. The last number used is called "the number of chairs (sheep, students, etc.)." Thus, there are seven stars here:

$$1\bigstar \quad 2\bigstar \quad 3\bigstar \quad 4\bigstar \quad 5\bigstar \quad 6\bigstar \quad 7\bigstar$$

Of course, since this process is so familiar to you, the preceding reminder of how to count is very boring, and counting, as such, *is* a boring task. Counting, however, is just *one* way to determine the number of objects in a collection. For instance, to find out how many stamps there are in a book of stamps, we would be foolish to count them all off, "one, two, three, ...," and so on. We could trust the number (if there is one) given on the cover, but if we wish to check for ourselves, we would probably proceed like this: Each page consists of a rectangular array of stamps; so we count the number of rows and the number of columns, then multiply these two numbers to determine the number of stamps on a page. Leafing through the book, we notice (*not* by counting!) that each page contains the same number of stamps. Last of all, we count the number of pages, multiply this by the number of stamps on a page, and so determine the number of stamps in the book.

Let us introduce an idea you have probably never considered, because it is at best a trivial notion. We ask, "How many eggs are in a dozen?" You properly answer "12." Since 12 answers a "how many" question, we call 12 a cardinal number. Likewise, we ask, "How many U.S. senators are citizens of Afghanistan?" You properly answer "none" or "0." Since 0 also answers a "how many" question, it is standard to call 0 a cardinal number. Thus, we take the cardinal numbers to be the set of numbers

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots .$$

And now, let us make some simple remarks about the cardinal numbers:

1. There is a first, or smallest, cardinal number: 0.
2. For each cardinal number, there is a unique next one, obtained simply by adding 1.
3. Remark 2 implies that there is no last, or greatest, cardinal number. That is the meaning of those three dots used in the above display of the cardinal numbers.

This last remark is worth some thought right now. In the first place, it means that we have enough cardinal numbers to count very large collections of objects and that—theoretically—we can count the number of grains of sand in the deserts of New Mexico or on the beaches of Florida. But even more, Remark 3 indicates how enormous the task of trying to study and discover properties of the cardinal numbers will be.

In the dunes of the White Sands National Monument, there are no more than 500,000,000,000,000,000 grains of sand; but most cardinal numbers (all but 500,000,000,000,000,001 of them) are greater than 500,000,000,000,000,000.

Certain cardinal numbers are called **digits**. The word "digit" comes from a Latin word meaning finger, and since we have ten fingers, our system of naming the cardinal numbers is based on the first ten cardinal numbers: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. These are the digits. You will easily recognize that every cardinal number is named by a string of digits, making the digits quite basic in our system of counting.

There is no word in the English language that forcefully expresses the general idea of *figuring out* how many things there are, as opposed to counting, "one, two, three, ..." We must use the word "counting" to express both ideas. Mostly, we will be using the words "count" and "counting" in the broader sense of determining "how many."

Problems

1. To count the number of students in a room, is it correct to count the number of males, then count the number of females, and then add the results?

2. To count the number of students in a college classroom, is it correct to count the number of people under 21 years of age, then count the number of people over 22 years of age, and then add the results?

3. To count the number of students in a college classroom, is it correct to count the number of people who are at most 21 years of age, then to count the number of people who are at least 21 years of age, and then add the results?

4. Repeat Problems 2 and 3 for a typical elementary schoolroom setting, still using the age 21.

5. In your discussion of Problem 4, might it be necessary to consider 0 as a cardinal number?

6. From your answers to Problems 2 and 3, you should be able to formulate a method for counting college students in a classroom—a method centering around the idea of 21 years of age. Do so.

7. A cardinal number is called **even** if it is twice some cardinal number; otherwise it is called **odd**. Thus, $26 = 2 \cdot 13$ and $0 = 2 \cdot 0$ are even, while 47 is odd, since 47 is not twice a cardinal number.

(a) Count how many odd digits there are. List them.

(b) Count how many even digits there are. List them.

(c) How many digits are either odd or even?

(d) How many digits are both odd and even?

8. A number is called "a multiple of 3" if it is three times a cardinal number.

   (a) How many digits *are not* multiples of 3? List them.

   (b) How many digits *are* multiples of 3? List them.

9. The counting number 32 is "made up" of the digits 3 and 2. But then so is 23, and 32 is certainly not the same as 23. Can you explain this?

10. The arrangement

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

is one arrangement of the digits, while

$$1, 5, 0, 7, 4, 6, 9, 8, 3, 2$$

is another. How many arrangements are there in all? *Don't* try to list them all, but do you think you could guess? Are there more than a million?

11. John learned to start counting with the number 5. At what age did he claim 15 candles on his birthday cake?

12. Bob is fond of the number 138; so when he counts he starts off with 138 and continues, $139, 140, 141, \ldots$, instead of starting with 1 and continuing, $2, 3, 4, \ldots$. If Bob counted 200 people in a room, how many would there actually be?

13. Eldon thinks a digit is *nifty* if it can be expressed as the sum of two distinct digits, neither of which is 0. In Eldon's thinking, how many nifty digits are there?

14. Just to be different from Eldon, Cletus decides to think of a digit as being nifty if it can be expressed as the *product* of two distinct digits, neither of which is 1. For Cletus, how many digits are nifty?

15. From the previous two problems, how many digits are nifty for both Eldon and Cletus? How many digits are nifty for neither?

1.2 Some counting problems and estimates

Now that you've been reminded of how simple it is to count, we would like to pose some questions for your consideration. They all involve counting, but the answers—though fairly easily obtained—are not easily computed by counting in the straight "one, two, three, ..." manner. By the time you finish this chapter, you will be able to answer these (and even much harder) questions:

1. How many different committees of five people each can be chosen from the United States Supreme Court? (Two committees are the same if and only if they are made up of the same group of five people.)

2. How many different committees of five people each can be chosen from the United States Senate?

3. Suppose a room contains seven men and five women and a circular table with six chairs. How many different seating arrangements are possible? How is this problem different from that of seating six people at a time *in a row*?

4. How many of the seating arrangements of Question 3 involve the seating of three men and three women, with the sexes alternating in the arrangement?

5. In a certain state, automobile license plates consist of either two or three letters, followed by three or four digits. How many different license plates are possible in that state?

6. In the American League Central Division baseball standings, how many outcomes are possible without ties, so that one team is in first place, one in second place, one in third place, one in fourth place, and one in fifth place (there being five teams in all)?

It is always desirable to have a rough estimate of what a number should be before going through a detailed computation to find out precisely what it is.

In fact, if you become adept at estimation, you can often use your estimate and forego any computation at all. Such "ballpark" estimates are constantly used in all phases of life to eliminate unfeasible plans. If you cannot read more than 400 words per minute, it will matter little to you whether a book contains 49,526 words or 50,324 words as far as the problem of reading it in an hour is concerned.

Try to guess the answers to the preceding questions. Even the broadest guesses will be good for a starter; for example, do you think the answer to Question 1 is closest to a hundred, a thousand, ten thousand, a hundred thousand, a million, ten million, a hundred million, or a billion? Try this sort of guessing for *each* of the six questions. In fact, before proceeding, record your answers to see how your guesses compare with the answers to be given shortly.

One purpose of guessing the answers to these questions is to convince yourself that counting can be difficult. From the size of some of the answers alone, you will realize that by trying a "one, two, three, ..." approach, you could not obtain the answer in your lifetime. (To drive the point home, find the answers to the following questions. How long is one million seconds? A hundred million seconds? A billion seconds? Would you like to spend that much time on one homework problem?) Another purpose is to demonstrate that intuition alone may not be adequate to obtain even an approximate answer. Third, you will appreciate more the fact that each of the exact answers can be obtained in about a minute, once some fundamental ideas about counting are understood.

The proper broad guesses are (1) a hundred, (2) a hundred million, (3) a hundred thousand, (4) a thousand, (5) a hundred million, and (6) a hundred. How did you do in your guessing?

The exact answers are (1) 126, (2) 75,287,520, (3) 110,880, (4) 4,200, (5) 200,772,000, and (6) 120. We repeat: Each of these answers was obtained by hand in about a minute, and one of the purposes of this chapter is to enable you eventually to master the fairly simple techniques used to arrive at those answers.

So far, we've done next to nothing, for you've been presented with counting problems too involved for the straightforward "one, two, three, etc." analysis. To prepare for the next section, here are some problems you can work out with a bit of thought and not too much time by the "one, two, three, ..." process. Later on, you will work such problems more easily.

Problems

1. The comic strip character Cathy has one pink, one blue, and one red pair of pants. She has one white blouse and one yellow blouse. List all the different outfits this gives her. How many are there?

2. One arrangement of the first three notes of the musical scale is

   do, re, mi.

   Another arrangement is

   re, do, mi.

   Write down all such arrangements. How many are there?

3. Write down all three-digit numbers consisting of the digits 2, 6, and 9. How many are there?

4. A popular hangout for young people in about 1930 might well have been called Ice-Cream-O-Rama, offering vanilla, chocolate, and strawberry ice cream—and three dips at that! Thus, one might order (from the bottom up) a dip of vanilla, a dip of chocolate, and a dip of vanilla; or one might order three dips of strawberry; or one might order two dips of vanilla and a dip of chocolate (this being counted as different from the first example given above, for people were very discriminating in 1930 and cared a lot about their ice cream).

   (a) How many different kinds of three-dip cones could you order from Ice-Cream-O-Rama?

   (b) We are not so discriminating today as they were in 1930, and we consider vanilla–chocolate–vanilla the same as vanilla–vanilla–chocolate. How many different kinds of three-dip cones could we order from Ice-Cream-O-Rama today?

5. Sam has some red paint, some yellow paint, and some green paint. He wants to paint the three windows in his room. If he uses just one color on each window, how many ways can he do it?

6. Little Frank has a penny, a nickel, a dime, a quarter, and a half-dollar. He has two pockets. In how many ways can he carry his money in his pockets?

7. Big Jack gave Little Frank a silver dollar. Now how many ways does Little Frank have of carrying his money? Can you use your result from the previous problem to help you figure this out?

8. How many different cardinal numbers are less than 100? Which is the smallest? Which is the largest?

9. Do you think you could solve Problem 10 of Section 1 by a listing process? (You might bother to try it awhile to see how futile it is.)

10. If all the people in the United States were laid end to end, would they reach around the world?

11. If all the people in the United States were dumped into the Grand Canyon, would they fill it up?

12. About how many years is one billion seconds?

13. About how many seconds do you expect to live?

1.3 A fundamental counting principle

The problems of counting—or at least some of them—will now be attacked in an orderly fashion. Throughout this and succeeding sections, we will use over and over again a principle of counting that we hope to make clear here. As you work through the examples that follow, you will come to understand an easier way to work the problems of Section 2 long before you will be able to compute those complex-looking answers in the body of Section 2.

Let us look at a problem. A Chicago family by the name of Kruse wants to travel to either New Orleans or Charleston for a vacation. Along the way (they'll be driving), they will stop over and visit friends in just one of these places: Dallas, Hattiesburg, Louisville, or Philadelphia. This means they *could* go first to Dallas and then to New Orleans; or they might decide to go to Charleston by way of Louisville. To view the various possibilities (and can you guess how many there are?), Mr. Kruse decides to make a map, so to speak. It looks something like this: