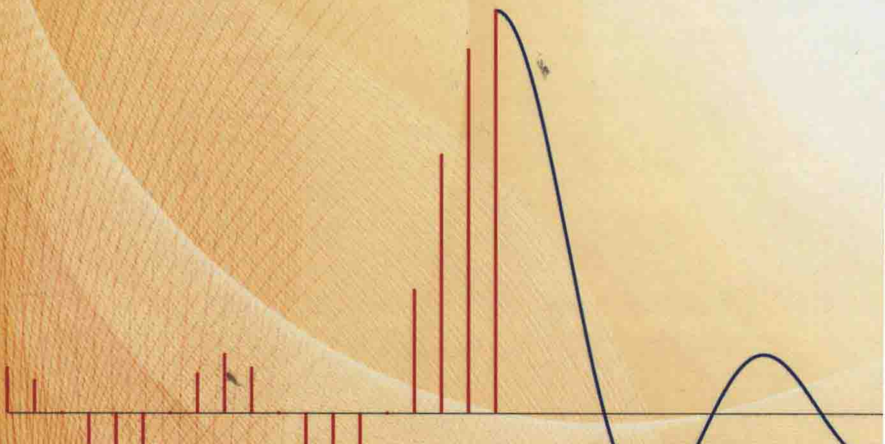
FOURIER TRANSFORMS

Principles and Applications

ERIC W. HANSEN



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To my family

PREFACE

The mathematical techniques known as "transform methods" have long been a basic tool in several branches of engineering and science, and no wonder. Fourier's simple idea, radical in its time, that a function can be expressed as a sum of sine waves, is ubiquitous. It underlies fields as diverse as communications, signal and image processing, control theory, electromagnetics, and acoustics. Electrical engineers typically encounter the rudiments of Fourier transforms in undergraduate systems and circuits courses, for modeling the spectral content of signals and designing frequency selective circuits (filters). The Laplace transform, a close cousin of the Fourier transform, enables the efficient analytical solution of ordinary differential equations and leads to the popular "S plane" and "root locus" methods for analyzing linear systems and designing feedback controllers. Discrete-time versions of the Fourier and Laplace transforms model spectra and frequency responses for digital signal processing and communications. Physics and engineering students meet the Fourier series when learning about harmonic motion or solving partial differential equations, for example, for waves and diffusion. The Fourier transform also models wave propagation from acoustics to radio frequencies to optics to X-ray diffraction. The widespread dissemination of the fast Fourier transform algorithm following its publication in 1965 added a computational dimension to all of these applications, from everyday consumer electronics to sophisticated medical imaging devices.

My purpose in writing this textbook is to pull these threads together and present a unified development of Fourier and related transforms for seniors and graduate students in engineering and physics—one that will deepen their grasp of how and why the methods work, enable greater understanding of the application areas, and perhaps motivate further pursuit of the mathematics in its own right. Drafts of the book have been used by myself and others for a 10-week course in Fourier transforms and complex variable theory at the Thayer School of Engineering, Dartmouth College. The prerequisites are an introductory course in lumped parameter systems (including the Laplace transform) or differential equations. Our course is itself prerequisite to courses in signal and image processing and a more advanced course in applied analysis taken by engineering and physics graduate students.

Philosophy and Distinctives

The book is more mathematically detailed and general in scope than a sophomore or junior level signals and systems text, more focused than a survey of mathematical methods, and less rigorous than would be appropriate for students of advanced mathematics. In brief, here is the approach I have taken.

1. The four types of Fourier transform on discrete and continuous domains—discrete Fourier transform (DFT), Fourier series, discrete-time Fourier transform (DTFT), and Fourier transform—are developed as orthogonal expansions within a vector space framework. They are introduced sequentially, starting with the DFT and working up to the continuous-time Fourier transform. The same important properties and theorems are revisited for each transform in turn, reinforcing the basic ideas as each new transform is introduced. This is in contrast with an approach that either begins with the continuous-time transform and works down to the others as special cases, or develops all four in parallel.
2. The early presentation of the DFT makes it immediately available as a tool for computing numerical approximations to the Fourier series and Fourier transform. Several homework problems give the student practice using numerical tools. MATLAB® is used throughout the book to demonstrate numerical methods and to visualize important ideas, but whether to use MATLAB or some other computational tool in the course is up to the instructor.
3. The fundamentals of complex analysis and integration are included as a bridge to a more thorough understanding of the Laplace and Z transforms, and as an additional way to calculate Fourier transforms.
4. Physical interpretations and applications are emphasized in the examples and homework problems. My hope is that the student will cultivate intuition for how the mathematics work as well as gain proficiency with calculation and application.
5. Starred sections, which may be skipped on a first reading, give brief introductions to more advanced topics and references for further reading.
6. Each chapter has a table of key results, which should be particularly helpful for reference after the course is completed.

Any author of an applied mathematics book must decide the extent to which the development of the material will be supported by proofs. The level of rigor required by a mathematician generally exceeds what is needed to justify the trustworthiness of a result to an engineer. Moreover, to prove all the key theorems of Fourier analysis requires a facility with real analysis and even functional analysis that exceeds the usual mathematical background of an undergraduate engineer. The approach taken here, for the most part, is to include proofs when they build intuition about how the mathematics work or contribute to the student's ability to make calculations and apply the transforms. Otherwise, I will usually substitute informal plausibility arguments, derivations of weaker results either in the text or in the end-of-chapter problems, or computational illustrations of the principles involved. Footnotes refer the interested reader to detailed treatments in more advanced texts.

Flow of the Book

The book has 10 chapters, which are described briefly here to show how the book's main ideas are developed. *Chapter 1* is a review of the topics from geometry, trigonometry, matrix algebra, and calculus that are needed for this course. *Chapter 2* then develops some fundamentals of vector spaces, particularly the generalizations of the geometric ideas of norm, inner product, orthogonality and orthogonal expansion from vectors to functions. This provides the unifying framework for the Fourier family and acquaints the student with concepts of broad importance in engineering mathematics.

Chapters 3–5 introduce, in sequence, the DFT, the Fourier series, the DTFT, and the (continuous-time) Fourier transform. The DFT has the easiest vector space interpretation of the four transforms, since it expands finite-dimensional vectors in terms of orthogonal finite-dimensional vectors. Some basic Fourier theorems (linearity, shift, energy conservation, convolution) are first presented here, then reappear later for the other transforms. Chapter 3 includes a derivation of the fast Fourier transform (FFT) algorithm and the discrete cosine transform (DCT), a close relative of the DFT that is the mathematical foundation of JPEG image compression.

The Fourier series, *Chapter 4*, is a representation of a periodic function as an infinite series of orthogonal sines and cosines. The appearance of the infinite series raises the question of convergence and leads to the important connections among convergence of the series, asymptotic behavior of the spectrum, and smoothness of the original function. The chapter includes applications to the diffusion and wave equations and to antenna arrays, and shows how to use the DFT to compute Fourier coefficients and partial sums of Fourier series. Swapping the time and frequency domains, the Fourier series becomes the DTFT, the basic tool for discrete-time system analysis and signal processing.

The Fourier transform, *Chapter 5*, expands an aperiodic function as a continuum of orthogonal sines and cosines rather than a set of discrete oscillatory modes. Despite the additional mathematical complication, it has many of the same properties as the DFT and the Fourier series and intuition developed earlier for these transforms will carry over to the Fourier transform. The chapter emphasizes using Fourier theorems for modeling systems (impulse response and transfer function) and performing calculations. It also shows how to use the DFT to compute transforms and convolutions. A brief introduction to time-frequency transforms and wavelet transforms concludes the chapter.

Chapter 6 begins by placing the impulse (delta) function on a more secure footing than the informal notion of "infinite height, zero width, unit area" that students sometimes bring with them from earlier classes. This is followed by development of a common, generalized framework for understanding ordinary functions together with impulses and other singular functions. Sampling theory, introduced informally in Chapter 3, is studied here in depth. It is also used to unify the four Fourier transforms, via the observation that sampling in the time domain produces periodicity in the frequency domain, and vice versa.

Chapters 7 and 8 are devoted to the theory of complex functions and methods of complex integration, with a focus on ultimately applying the theory to understanding and calculating transforms. Numerical calculations of the basic complex integral $\oint z^n \, dz$ on different closed contours are used to help students visualize why the integral evaluates either to $2\pi i$ or to zero, before formally introducing the fundamental results, the Cauchy integral theorem and integral formula. The traditional subjects of conformal mapping and potential theory are omitted, but the complex variable introduction here is, I believe, sufficient preparation for subsequent courses, for example, in electromagnetism or fluid mechanics, where complex potentials may be useful.

Chapter 9 moves beyond the Fourier transform to the Laplace, Z, and Hilbert transforms. The Laplace transform is motivated by the need to handle functions, in particular ones that grow exponentially, that are beyond the reach of the Fourier transform. The familiar Laplace theorems, used to solve initial value problems, are derived and compared with their Fourier counterparts. The well-known partial fraction expansion method for Laplace inversion is connected with complex integration and extended beyond the rational functions encountered in linear system theory. The Z transform appears via the Laplace transform of a sampled function, and analogies between the transforms are emphasized. The Hilbert transform, which describes a special property of the Fourier transform of a one-sided function, is developed and applied to various problems in signal theory.

Chapter 10 concludes the book by revisiting the Fourier transform in two and three dimensions, with applications to wave propagation and imaging. The closely related Hankel and Radon transforms are introduced. Multidimensional transforms of arrays of impulses are developed and applied to sampling theory and X-ray crystallography.

Suggested Use

Most of Chapters 2–5, 7 and 8, and selected parts of Chapters 6 and 9, are covered in my 10-week (30-hour) course. In a full semester course, additional material from Chapters 6, 9, and 10 could be added. If students have already had a course in complex analysis, or if time does not permit, Chapters 7 and 8 may be skipped, with the caveat that portions of Chapter 9 are inaccessible without complex integration. However, this would permit a thorough coverage of Chapters 2–6 and 10 with selected topics from Chapter 9. While I naturally prefer the sequence of Chapters 2–4, they may be approached in a different order, with the Fourier series before the DFT, and with the vector space material presented "just in time" as the Fourier methods are introduced. End-of-chapter problems cover basic and more complex calculations, drawn from the theory itself and from many physical applications. I hope that instructors will find sufficient variety to suit their particular approaches to the material.

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Eric W. Hansen

Hanover, New Hampshire
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