



# JUNIOR HIGH SCHOOL MATHEMATICS

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GINN AND COMPANY

BOSTON • NEW YORK • CHICAGO • LONDON  
ATLANTA • DALLAS • COLUMBUS • SAN FRANCISCO

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**627.7**

**The Athenæum Press**

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## PREFACE

The gratifying development of the junior high school in this country has given rise in recent years to two important lines of investigation. The first concerns itself with the problems of the curriculum, and the second with the improvement of the quality of instruction. Each has raised a series of questions in the minds of teachers and administrators, and upon the answers that shall be given depends in a large degree the success or failure of this new type of school. Inspired to a considerable extent by the work of the International Commission on the Teaching of Mathematics, by the report of the National Committee appointed by the Mathematical Association of America, and by the results of the testing movement, teachers have been seeking these answers and have, through local, state, and national organizations, seriously studied the problems that have arisen. The result has been a body of conclusions that deserve the attention of the schools and of the public at large.

In this work the authors have set forth the leading problems that have been considered, the conclusions generally reached, the questions needing further study, and some of the means that tend to make this study more effective. They have also made various suggestions for improving the curriculum, the methods of instruction, and the testing of pupils; for selecting the objectives in each of the important branches of mathematics in the junior high school; for making the subject seem more real and interesting; and for the treatment of various other topics as set forth in the table of contents.

If properly organized and carefully presented, no subject in the curriculum should ever seem dull or uninteresting. Even if a retarded group is unable to keep pace with the rest of the class, the work should be so planned as to appeal to its members and lie within their ready grasp. The same is true with respect to the more brilliant pupils, — often the most retarded in op-

portunity to advance; their work should be interesting to their higher intelligence and should be worthy of their superior powers. It is a sad confession to make that our standard of teaching is too often set by the demands of mediocrity. It is one of the purposes of this book to show how the curriculum can be prepared to meet the needs of three general types of pupils, — the two extremes referred to above and the large middle group commonly known as the average. The administrative problem is by no means easy of solution in the smaller schools, and hence the results will naturally depend largely upon local conditions and upon the needs and abilities of each school system.

Our present problem is, however, something more than a mere listing of objectives, the stating of minimum essentials, or the differentiating of the material within our courses. It is largely a matter of developing better teachers, — teachers of improved and enlarged academic training in mathematics, who are versed in the psychology of childhood and are possessed of such a love for children as to make the work of teaching a delight. Given the most ideal curriculum and the most promising pupils possible, still with a "humdrum" teacher who lacks the elements mentioned the work is bound to be uninspiring and the results unsatisfactory. No book, however stimulating, can wholly remodel such a teacher or make up for inborn defects. On the other hand, for those who give real promise of success in teaching mathematics it is hoped that the message herein contained will be helpful.

Since the book will be used chiefly in institutions of collegiate grade and by those who are preparing for positions of more than usual responsibility, there have been included numerous bibliographies and lists of questions and topics for discussion. These will, it is believed, serve as useful aids to instruction. The text itself, however, has been prepared quite as much for the use of the individual teacher who has no opportunity to attend college classes as for those who, in increasing numbers, frequent our summer schools and university courses.

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# THE TEACHING OF JUNIOR HIGH SCHOOL MATHEMATICS

## CHAPTER I

### HOW THE CURRICULUM IS DETERMINED

#### 1. DEFINING THE OBJECTIVES

**Need for Objectives.** In teaching mathematics, as in any other kind of work we set out to do, we cannot expect to realize our aims unless they are precise and are clearly defined at the outset. The first step, therefore, in planning a modern course of study in mathematics for the junior high school is to prepare a list of desirable objectives which we hope to attain. Such a list should not be imposed upon the schools by some higher authority; on the contrary, it should be the result of much thought and discussion on the part of those who will actually use it, who know the work of the classroom, and who themselves are the authority as to what can be expected of children. If properly made and carefully checked, a list of this kind is of value, not merely to the inexperienced teacher but to those who have already become accustomed to consider the constantly changing problems of the proper selection of material.

**Nature of Objectives.** There are two kinds of objectives to be kept in mind,—the great central ones and those that are more specifically mathematical. The former have to do with the realization of aims that are not peculiar to mathematics alone but which are to be sought in all fields of knowledge. Thus, the central objective in teaching junior-high-school mathematics is to develop well-educated citizens. This purpose is much the same in the teaching of all the great departments of knowledge, such as literature, history, language, art, mathematics, natural

science, religion, and contemporary civilization. Everyone has a right to know the general nature of each of these departments. Mathematics being one of them, a strictly mathematical objective is to give the pupils in the junior high school some idea of the general nature and uses of both business and social arithmetic, of intuitive geometry, of practical algebra, and of numerical trigonometry of a simple type, and some knowledge of the meaning of a demonstration in geometry. In other words, we should give as complete and all-round a view of the nature of mathematics as possible in the time allowed.

**Function of the Junior High School.** The junior high school should therefore offer an opportunity for making a general survey of the meaning of the great branches of human knowledge. It should not fear to be superficial, for its mission is to show a wide surface rather than a narrow depth. It should be at the same time a period in which interest should be awakened, knowledge extended, and a diagnosis made of the pupil's abilities and tastes. At its close each pupil should have shown his parents, his teachers, and himself, in a rather general way, what is his natural bent of mind and what it will very likely continue to be. If he has no taste for or ability with respect to mathematics, and does not expect to enter a college or a technical school, he should no longer be required to pursue the subject, and similarly with respect to other important branches. His mind has been given its chance; it should now be given a reasonable choice, — a matter to be determined by his parents, his teachers, and himself. It is not likely to be such as to lead to any narrow specialization in the senior high school, but it is quite possible that it will lead him to cease studying certain special subjects, — say natural science, foreign language, mathematics, drawing, or the manual arts, — for which he has no taste and in which he has shown no ability. This does not mean unrestrained license of election or freedom to choose "snap courses"; it simply means that there is little use in continuing in a field in which there is neither interest nor hope, or in attempting to proceed to college when the mind does not permit of laying the necessary foundations.

**Classification of Mathematical Objectives.** For convenience, mathematical objectives also may be classified in two groups, — the great central mathematical objectives and the more specific mathematical objectives, the latter being subsidiary in the entire scheme. These objectives may be listed in various ways; they may be placed in alphabetical sequence like the words in a dictionary, they may be arranged according to their difficulty for the pupils, they may be classified according to the branches of mathematics commonly taught, they may be set forth with little or no regard to pedagogical sequence, or they may be given in the order which experience has shown to be the most helpful to teachers, even though there is some sacrifice of logical sequence and some duplication of details. In this discussion, as well as in the more elaborate treatment in Chapter III, we shall follow the last of these five plans.

**List not Final or Complete.** As will later be stated in Chapter III, where the matter will be more fully discussed, the lists of objectives should not be considered as either final or complete; in fact, in this case, it is purposely left incomplete. To submit a longer list at present might be wearisome and discouraging and thus do more harm than good to teachers who are trying to clarify their views. On the other hand, too short a list might mean that the pupils throughout the country would not have the advantage of a proper survey of the possible field. The teacher who feels so inclined and has reasonable confidence in himself should modify the list to meet the needs of the community which he is to serve, keeping in mind that some things in the course should be uniformly taught everywhere. In making the selection emphasis should be placed upon the fact that everything included should be good material; in other words, the objectives should be fundamental.

**Basis of Choosing Objectives.** The objectives selected have been chosen after a careful consideration of three important criteria: (1) the intrinsic worth of mathematics itself; (2) the social needs of people in general, and especially those in the local community; and (3) the interests of the pupils who are to study the subject. The first criterion requires that the

teacher must know mathematics as thoroughly as possible in order to judge the value of its various branches. In any case, he owes it to himself and to his pupils to increase his knowledge continually. The second criterion means that the future well-being and social status of the nation and community must be considered with respect to what mathematics will be useful to the average, well-educated citizen. The third criterion means that the objectives must be such that the material chosen for each grade shall be that which is likely to be the most valuable to the pupil if he should leave school at the end of that year. Moreover, it is important that the pupil be led to see the real worth of the objectives chosen in order that the chance of realizing them may thus be increased.

## 2. SELECTION OF ACTIVITIES AND MATERIAL

**Choice of Subject Matter.** In the past, various methods of choosing the subject matter of mathematics have been used. Although much good material has been selected by each of these methods, most if not all of them have their disadvantages. The method of selecting material from various courses of study in use throughout the country, although representative of what is being taught, is open to serious objection, it being well known that such courses generally tend to perpetuate obsolete processes and antiquated business methods, and usually fail to be of help in suggesting the thing which ought to be taught.

It is equally true that the best material cannot be secured by making an inventory of the current textbooks in mathematics. They too are frequently guilty of overemphasizing unimportant or obsolete material. It is also true that textbook writers are not always able to suggest newer and better things, and that textbooks are often made merely to fit the demands of certain state syllabuses which contain much obsolete and otherwise undesirable material. It is, of course, true that for many schools the textbook is and will continue to be the curriculum, but this does not obviate the necessity for pioneer work on the part of

progressive teachers and for freedom to supplement or rearrange the material in a book that is reactionary in treatment, is insufficient in its offering, or admits of improvement in sequence.

The makers of standardized tests in recent years have erred in including in their work certain exercises and problems that thoughtful teachers everywhere have no desire to see perpetuated in our schools. In fact many of these undesirable elements were obtained by the makers of tests from existing courses of study and from textbooks. Thus it is obvious that such tests cannot be used as the sole basis in the selection of desirable material.

We also know that it is not safe to try to determine what mathematics should be taught by merely counting the frequency with which certain mathematical terms are used in a few current editions of newspapers and magazines. Such a method is so unreliable that even those who have pretended to believe in it are abandoning it for more reliable criteria.

Finally, it is fair to say that we cannot satisfactorily formulate the course of study by going out in the world and asking individuals chosen at random what mathematics is useful to them. The fact is that not one of them ever knows just what use he has made of mathematics. Moreover, no one of them has probably given thought to the question of determining how he might have used mathematics profitably if he had only given the matter a little serious attention.

Any one of the above criteria may be of service in selecting the material for a course, but not one or even all will be sufficient for our purpose. If the objectives selected are to meet our modern needs, we must have at least one other criterion, — one that is at the same time the result of experience and of good judgment. This last criterion is the opinion of the most expert among the well-trained teachers of mathematics — those who are able not only to tell how they use the science but also to show how it may be used in the present and in the future for the betterment of mankind.

**Method of Making the Selection.** When people have to consult with experts, they usually get the opinion of the best they

can find. The material suggested in the list given later, in Chapter III, is the result of the combined judgment of many different teachers who not only know what is taught in the schools, but who are also the best qualified to speak with authority as to the needs of the new curriculum in mathematics in the training of the educated citizen. They are teachers of wide experience, themselves often trainers of teachers, and in several cases those to whom the latter have come for further preparation in their work. The importance of securing such a consensus of opinion cannot be overemphasized. To reach all such teachers is manifestly impossible; to reach carefully selected groups has been the purpose attained.

**Importance of a Flexible Curriculum.** It is needless to say that the material of the curriculum should not be allowed to crystallize too soon, if at all. It is much more important to keep it plastic and subject to the possibility of improvement. In other words, if properly conceived, curriculum construction is a continuous process. It is therefore important that we encourage careful discussion of all suggested plans for reorganization, that we weigh the opinions of the best writers, and that we avoid any spasmodic adoption of some suggested scheme as our final decision upon this important problem.

### 3. THE PLACE OF METHOD

**Importance of Method.** A Latin writer long ago remarked, "Poets are born, not made." In a way the same is true of teachers and, indeed, of those who succeed in any other walk of life. But however valuable inherent qualities of native ability may be, mere knowledge does not suffice; mere "knack" will not serve the purpose; the successful teacher must not merely be born, he must acquire knowledge of his subject and must profit by the counsel of experts in his chosen field. It is therefore important that we should consider with care the question of method.

**Two Phases of the Subject.** We should remember, however, that there are two sides to method, namely, methods of teaching

and methods of learning. We have done a great deal with the former in our normal schools and teachers' colleges, but it is only recently that we have begun to study seriously the psychology of learning. In later chapters we shall give more attention both to the methods of teaching and to the psychology of acquiring knowledge.

#### 4. THE PSYCHOLOGY OF LEARNING

**How the Pupil Learns.** The practical problem in the psychology of learning is to find out how pupils learn most easily and economically, and we cannot expect to establish the most satisfactory curriculum unless we know the important results of this investigation. It is useless to say that we wish to teach certain subject matter when we know that this is so difficult that the pupils cannot comprehend it, for the time required to teach such topics is out of all proportion to their importance in the education of the ordinary citizen.

**The Material for the Course.** The material which will be suggested later is such as can be taught to children in the junior high school in the time ordinarily allotted to mathematics. It is material that has been tested in the classroom and is a necessary part of the equipment of every citizen. The teacher, however, should adjust it to the needs of his local group, keeping in mind the fact, as already stated, that the fundamental topics selected should meet with general approval throughout the country.

**Proper Organization and Grade Placement.** What has just been said implies that the material in the course of study should be effectively organized, arranged in convenient units, and properly placed with respect to the grades in which it is to be used. If a particular city or school undertakes to establish a curriculum which is not in harmony with any textbook that it can find, — usually a doubtful policy, — only those who are qualified by superior training, scholarship, and experience should be admitted to the committee in charge of the work. With all the variety of textbooks available, it is generally pos-



sible to find one that can be followed to better advantage than any local program that can be devised. With respect, however, to correlating the work to local needs and interests, the work of a curriculum committee is valuable.

## 5. A TESTING PROGRAM

**The Place of Tests.** The last but by no means the least subject to be considered in determining the proper curriculum in mathematics is that of an adequate testing program. The purpose of such a program is to find by proper diagnostic tests whether the objectives previously stated are being fully realized. Without tests we cannot know whether the material is too difficult for the pupils. If the desired degree of mastery has not been secured, we wish to ascertain how good the results really are. The tests will reveal many things about the strengths and weaknesses of the pupils that cannot be as well discovered by any other means. In this way they become instruments for diagnosis and for the improvement of instruction.

**Remedial Work.** When such diagnostic tests are given and the results are poor, the teacher knows that his work was poor, that the pupils did not have the will to learn, or that the material was intrinsically too difficult. In any case, the teacher can use the results as a basis for such remedial teaching as he thinks necessary. If the pupils have not done their part, he is faced with a disciplinary problem; but if he decides that the material is too difficult, certain objectives may have to be modified.

**Nature of Tests.** No test can be condemned if it serves the purpose for which it is intended. We should be careful, however, to secure the best tests available and to use them to the best advantage. The greatest need in the matter of tests in the junior high school is for those which are diagnostic and which test what has been taught in the course. Such tests are also serviceable as teaching devices, supplementing the textbook, supplying material for drill upon the fundamental skills, and serving as measuring devices. A more complete discussion of the testing program will be given in Chapter XI.