COMPLEX FUNCTION THEORY

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COMPLEX FUNCTION THEORY

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PREFACE

τῷ παθεῖν μαθεῖν

This book is in two parts. The first is intended to serve as a basis for a first course on complex function theory for both undergraduate and graduate students. It is not expected that the whole of Part One will be "covered" in a semester. Experience and realism would say that the larger part of the first seven chapters is manageable. Part One can constitute a semester course when an adequate level of maturity is available. Chapter IX, the final chapter of Part One, culminates in the proof of the prime number theorem of Hadamard and de la Vallée Poussin—a particularly handsome application of topics treated in earlier chapters. I dare say that a syllabus consisting of the material presented in Part One forms a reasonable and, in fact, mathematically appealing program that can serve as a basis for the requirements in complex function theory for all doctoral candidates in mathematics at present.

Part Two seeks to supplement and round out the exposition for students who are to use complex function theory seriously in their professional work. It should serve as a foundation for subsequent courses in the area of complex analysis on such topics as the Nevanlinna theory, Riemann surfaces, several variable theory, and differential equations. It may be used for a proseminar in analysis for first- or second-semester graduate students; or taken in conjunction with Chs. VIII and IX it could serve as the basis of a second-semester course on complex function theory in which active student participation is a predominant note. I have envisaged the role of Part Two as twofold: to increase the student's experience with the ideas and methods of Part One and to exploit the applicability of these methods.

It is my intention to be specific about material from other disciplines which will be used in this book. In particular, this will be the case for the theorems of elementary real analysis, including the differential calculus. Precisely here more than anywhere else the instructor of a first-semester course on complex function theory encounters his greatest tribulations. The scope of elementary courses on real analysis is so varied that frequently students of excellent ability come to their first course on complex function theory with a less than rudimentary grasp of the basic and most elementary theorems of the one and several variable real differential calculus.

I have tried not to presume on the reader's background for material from other fields. Exceptions are: the representation of a positive integer as a product of primes in Ch. VIII for which see [14]; improper integrals in Chapters V, VII, IX, XIV for which see [32a]; and standard information from the elements of linear algebra in Chs. XII, XVIII for which see [58]. It is reasonable to suppose that the reader will be familiar with these questions.

The student should ultimately recognize that four principal methods dominate complex function theory, methods closely associated with the founders of the subject: the power series approach, the complex integral approach embracing the Cauchy theory in its full range, the approach based on the connection with the theory of harmonic functions, and the mapping approach. The first method has its roots in the work of the eighteenth century and was developed with grandeur by Weierstrass. The last two methods find their sources in the work of Gauss and Riemann. At the beginning of the study of complex function theory the Cauchy theory stands out because of its dazzling virtuosity.

The status of complex function theory has changed greatly during this century. At the beginning it had many triumphs. Today they are fewer. It is also characteristic that complex function theory does not appear in splendid isolation but rather along with other branches of mathematics as a component interrelating with other components of an organic whole. The proud vaunt of "rein funktionentheoretisch" belongs to an era that is past.

At the stage of the student's development when he encounters complex function theory, he will almost inevitably be subjected to bouts of rigor, to "rectitude with exactitude." It is in the nature of things and doubtless it would be a mistake to put aside such aspects of mathematical training. However, I hope the student will seek a light touch.

I wish to acknowledge the influence of the books of Carathéodory [21] and of Saks and Zygmund [108]. The book of Saks and Zygmund is a model for exposition of the highest standard which is free of refinement profitless at this level. Special thanks are due to Professor F. M. Stewart for having communicated to me earlier his treatment of the Cauchy theorem.

There remains for me to express my thanks to Professors Eilenberg and Smith for their long-standing invitation to write a volume on complex function theory for their series, to the University of Illinois for affording me the opportunity to write this book, and to Academic Press for its helpful cooperation.

January 1968
Urbana, Illinois

MAURICE HEINS

CONTENTS

Chapter III. Topological and Metric Spaces

Chapter IV. Complex Differential Analysis

Chapter V. Cauchy Theory

Chapter VI. Laurent Expansion. Meromorphic Functions

Chapter VII. Further Applications of the Cauchy Theory

Chapter VIII. The Zeros and Poles of Meromorphic Functions

Chapter IX. The Gamma and Zeta Functions. Prime Number Theorem

PART TWO

Chapter X. Supplementary Developments Concerning C

Chapter XI. Complex Differential Coefficients

Chapter XII. Topics in the Theory of Power Series

Chapter XIII. Harmonic and Subharmonic Functions

Chapter XIV. Complements to the Cauchy Theory

Chapter XV. Möbius Transformations

Chapter XVI. The Modular Function λ. The Picard Theorems

Chapter XVII. Some Questions Concerning Univalent Analytic Functions

Chapter XVIII. Riemann Surfaces

PART ONE

Chapter I

THE REAL FIELD

PREFATORY REMARKS

It is taken for granted that the user of this book has already studied elementary real analysis. Desiderata for the purposes of complex function theory are: the properties of the real number system, limit processes, sequences, continuity, the Riemann-Stieltjes integral, and the basic theorems of the differential calculus in one and several variables. However, there is considerable variation of opinion concerning what elementary real analysis consists of. Also, the actual amount of material treated in any given class is apt to be very variable even in the presence of a fairly standard syllabus. A period of adjustment between the reader and the book is probably inevitable. In this chapter we give a succinct summary of material that should by and large be well known to the student. The familiar parts should be read for review, attention being given to the exercises. It is my experience that the notions of *limit superior* and *limit inferior* are much less commonly known by students than instructors might think. Since these notions are of considerable use in complex function theory, they merit prompt attention.

For readers wishing supplementary material we suggest Ch. 1 of [133], Chs. 1 and 2 of [36], and Chs. 1, 2, and 3 of [107].

1. SETS

The usual material of elementary set theory is so current that we may certainly take it for granted. Thus, without explanation we use as known the