John Stillwell

Classical Topology and Combinatorial Group Theory

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Preface

In recent years, many students have been introduced to topology in high school mathematics. Having met the Möbius band, the seven bridges of Königsberg, Euler's polyhedron formula, and knots, the student is led to expect that these picturesque ideas will come to full flower in university topology courses. What a disappointment "undergraduate topology" proves to be! In most institutions it is either a service course for analysts, on abstract spaces, or else an introduction to homological algebra in which the only geometric activity is the completion of commutative diagrams. Pictures are kept to a minimum, and at the end the student still does not understand the simplest topological facts, such as the reason why knots exist.

In my opinion, a well-balanced introduction to topology should stress its intuitive geometric aspect, while admitting the legitimate interest that analysts and algebraists have in the subject. At any rate, this is the aim of the present book. In support of this view, I have followed the historical development where practicable, since it clearly shows the influence of geometric thought at all stages. This is *not* to claim that topology received its main impetus from geometric recreations like the seven bridges; rather, it resulted from the *visualization* of problems from other parts of mathematics—complex analysis (Riemann), mechanics (Poincaré), and group theory (Dehn). It is these connections to other parts of mathematics which make topology an important as well as a beautiful subject.

Another outcome of the historical approach is that one learns that classical (prior to 1914) ideas are still alive, and still being worked out. In fact, many simply stated problems in 2 and 3 dimensions remain unsolved. The development of topology in directions of greater generality, complexity, and abstractness in recent decades has tended to obscure this fact.

Attention is restricted to dimensions ≤ 3 in this book for the following reasons.

(1) The subject matter is close to concrete, physical experience.
(2) There is ample scope for analytic, geometric, and algebraic ideas.
(3) A variety of interesting problems can be constructively solved.
(4) Some equally interesting problems are still open.
(5) The combinatorial viewpoint is known to be completely general.

The significance of (5) is the following. Topology is ostensibly the study of arbitrary continuous functions. In reality, however, we can comprehend and manipulate only functions which relate finite "chunks" of space in a simple combinatorial manner, and topology originally developed on this basis. It turns out that for figures built from such chunks (simplexes) of dimension ≤ 3, the combinatorial relationships reflect all relationships which are topologically possible. Continuity is therefore a concept which can (and perhaps should) be eliminated, though of course some hard foundational work is required to achieve this.

I have not taken the purely combinatorial route in this book, since it would be difficult to improve on Reidemeister's classic *Einführung in die Kombinatorische Topologie* (1932), and in any case the relationship between the continuous and the discrete is extremely interesting. I have chosen the middle course of placing one combinatorial concept—the fundamental group—on a rigorous foundation, and using others such as the Euler characteristic only descriptively. Experts will note that this means abandoning most of homology theory, but this is easily justified by the saving of space and the relative uselessness of homology theory in dimensions ≤ 3. (Furthermore, textbooks on homology theory are already plentiful, compared with those on the fundamental group.)

Another reason for the emphasis on the fundamental group is that it is a two-way street between topology and algebra. Not only does group theory help to solve topological problems, but topology is of genuine help in group theory. This has to do with the fact that there is an underlying computational basis to both combinatorial topology and combinatorial group theory. The details are too intricate to be presented in this book, but the relevance of computation can be grasped by looking at topological problems from an algorithmic point of view. This was a key concern of early topologists and in recent times we have learned of the *nonexistence* of algorithms for certain topological problems, so it seems timely for a topology text to present what is known in this department.

The book has developed from a one-semester course given to fourth year students at Monash University, expanded to two-semester length. A purely combinatorial course in surface topology and group theory, similar to the one I originally gave, can be extracted from Chapters 1 and 2 and Sections 4.3, 5.2, 5.3, and 6.1. It would then be perfectly reasonable to spend a second semester deepening the foundations with Chapters 0 and 3 and going on to 3-manifolds in Chapters 6, 7, and 8. Certainly the reader is not obliged to master Chapter 0 before reading the rest of the book. Rather, it should be skimmed once and then referred to when needed later. Students who have had a conventional first course in topology may not need 0.1–0.3 at all.

The only prerequisites are some familiarity with elementary set theory, coordinate geometry and linear algebra, ε-δ arguments as in rigorous calculus, and the group concept.

The text has been divided into numbered sections which are small enough, it is hoped, to be easily digestible. This has also made it possible to dispense with some of the ceremony which usually surrounds definitions, theorems, and proofs. Definitions are signalled simply by italicizing the terms being defined, and they and proofs are not numbered, since the section number will serve to locate them and the section title indicates their content. Unless a result already has a name (for example, the Seifert–Van Kampen theorem) I have not given it one, but have just stated it and followed with the proof, which ends with the symbol □.

Because of the emphasis on historical development, there are frequent citations of both author and date, in the form: Poincaré 1904. Since either the author or the date may be operative in the sentence, the result is sometimes grammatically curious, but I hope the reader will excuse this in the interests of brevity. The frequency of citations is also the result of trying to give credit where credit is due, which I believe is just as appropriate in a textbook as in a research paper. Among the references which I would recommend as parallel or subsequent reading are Giblin 1977 (homology theory for surfaces), Moise 1977 (foundations for combinatorial 2- and 3-manifold theory), and Rolfsen 1976 (knot theory and 3-manifolds).

Exercises have been inserted in most sections, rather than being collected at the ends of chapters, in the hope that the reader will do an exercise more readily while his mind is still on the right track. If this is not sufficient prodding, some of the results from exercises are used in proofs.

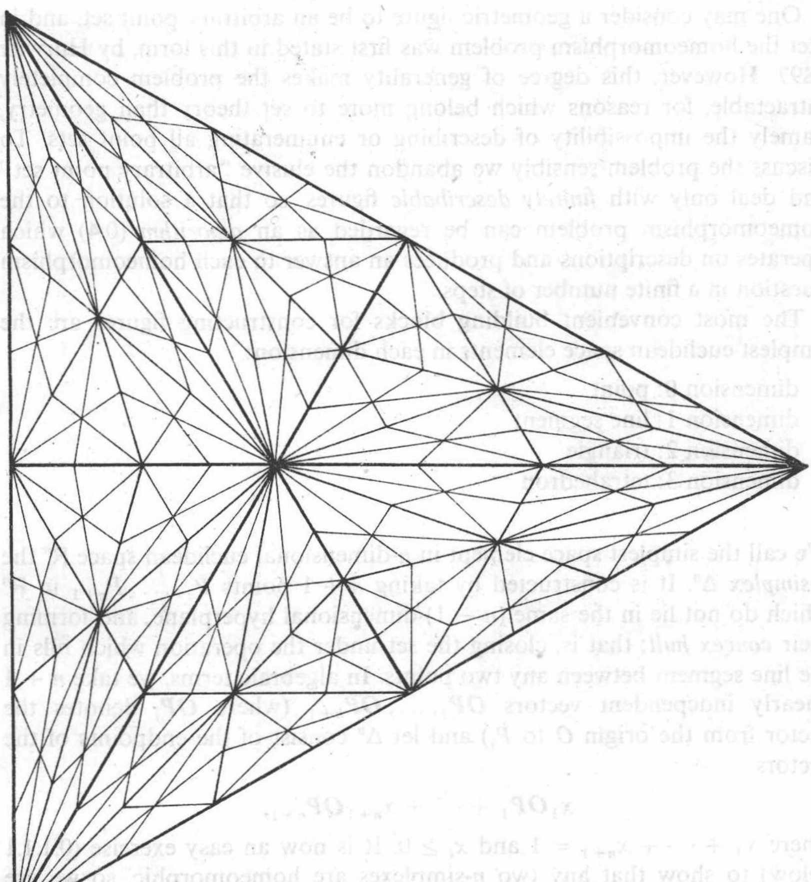
The text has been improved by the remarks of my students and from suggestions by Wilhelm Magnus and Raymond Lickorish, who read parts of earlier drafts and pointed out errors. I hope that few errors remain, but any that do are certainly my fault. I am also indebted to Anne-Marie Vandenberg for outstanding typing and layout of the original manuscript.

October 1980 JOHN C. STILLWELL

Contents

CHAPTER 0

Introduction and Foundations



0.1 The Fundamental Concepts and Problems of Topology

0.1.1 The Homeomorphism Problem

Topology is the branch of geometry which studies the properties of figures under arbitrary continuous transformations. Just as ordinary geometry considers two figures to be the same if each can be carried into the other by a rigid motion, topology considers two figures to be the same if each can be mapped onto the other by a one-to-one continuous function. Such figures are called topologically equivalent, or *homeomorphic*, and the problem of deciding whether two figures are homeomorphic is called the *homeomorphism problem.*

One may consider a geometric figure to be an arbitrary point set, and in fact the homeomorphism problem was first stated in this form, by Hurwitz 1897. However, this degree of generality makes the problem completely intractable, for reasons which belong more to set theory than geometry, namely the impossibility of describing or enumerating all point sets. To discuss the problem sensibly we abandon the elusive "arbitrary point set" and deal only with *finitely describable* figures, so that a solution to the homeomorphism problem can be regarded as an *algorithm* (0.4) which operates on descriptions and produces an answer to each homeomorphism question in a finite number of steps.

The most convenient building blocks for constructing figures are the simplest euclidean space elements in each dimension:

dimension 0: point
dimension 1: line segment
dimension 2: triangle
dimension 3: tetrahedron
⋮

We call the simplest space element in n-dimensional euclidean space $\mathbf{R}^n$ the *n-simplex* Δ^n. It is constructed by taking $n + 1$ points $P_1, \ldots, P_{n+1}$ in $\mathbf{R}^n$ which do not lie in the same $(n - 1)$-dimensional hyperplane, and forming their *convex hull*; that is, closing the set under the operation which fills in the line segment between any two points. In algebraic terms, we take $n + 1$ linearly independent vectors $\boldsymbol{OP}_1, \ldots, \boldsymbol{OP}_{n+1}$ (where $\boldsymbol{OP}_i$ denotes the vector from the origin O to P_i) and let Δ^n consist of the endpoints of the vectors

$$x_1\boldsymbol{OP}_1 + \cdots + x_{n+1}\boldsymbol{OP}_{n+1},$$

where $x_1 + \cdots + x_{n+1} = 1$ and $x_i \geq 0$. It is now an easy exercise (0.1.1.1 below) to show that any two n-simplexes are homeomorphic, so we are entitled to speak of *the* n-simplex Δ^n.

Each subset of $m + 1$ points from $\{P_1, \ldots, P_{n+1}\}$ similarly determines an *m-dimensional face* Δ^m of Δ^n. The union of the $(n - 1)$-dimensional faces is called the *boundary* of Δ^n, so all lower-dimensional faces lie in the boundary. We shall build figures, called *simplicial complexes*, by pasting together simplexes so that faces of a given dimension are either disjoint or coincide completely. This method of construction, which is due to Poincaré 1899, will be studied more thoroughly in 0.2. For the moment we wish to claim that all "natural" geometric figures are either simplicial complexes or homeomorphic to them, which is just as good for topological purposes.

This claim is supported by some figures which play a prominent role in this book—surfaces and knots. Surfaces may be constructed by pasting triangles together, so they are simplicial complexes of dimension 2. For example, the surface of a tetrahedron (which is homeomorphic to a sphere) is a simplicial complex of four triangles as shown in Figure 1. The torus surface (Figure 2) can be represented as a simplicial complex as shown in Figure 3. The representation is of course not unique, and from this one begins to see the *combinatorial* core of the homeomorphism problem, which remains after the point set difficulties have been set aside. Given a description of a surface as a list of triangles and their edges, how does one assess its *global* form? In particular, are the sphere and the torus topologically different? In fact we know how to solve this problem (by the classification theorem of 1.3, and 5.3.3), but not the corresponding 3-dimensional problem.

Much of the difficulty in dimension 3 is due to the existence of *knots*. We could define a knot to be any simple closed curve $\mathscr{K}$ in $\mathbf{R}^3$, but any such

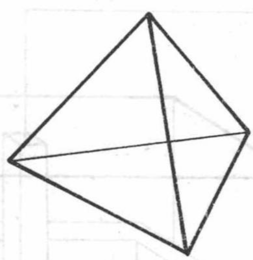


Figure 1

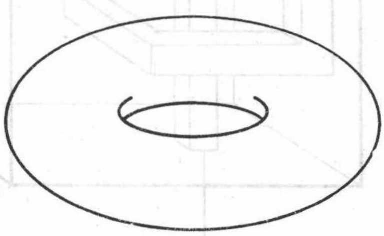


Figure 2

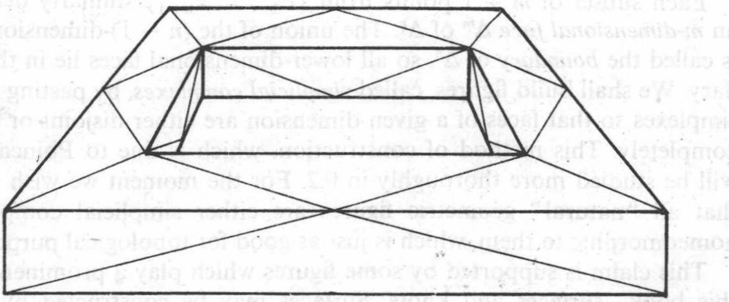


Figure 3

$\mathcal{K}$ is homeomorphic to a circle and its "knottedness" actually resides in the complement space $\mathsf{R}^3 - \mathcal{K}$. This space is not finitely describable in terms of simplexes, so we replace R^3 by, say, a cube and drill a thin tube out of it following the "knotted part" of $\mathcal{K}$ (see Figure 4).

This figure can be divided into small tetrahedra and hence is a finite simplicial complex representing the knot. The homeomorphism problem for such figures is extraordinarily difficult; Riemann was perhaps the first to think about it seriously (see Weil 1979), and it has been solved only recently (see Hemion 1979, Waldhausen 1978). The solution extends to more general "knot spaces" obtained by drilling any number of tubes out of cubes, but not as yet to all the figures which result from pasting knot spaces together.

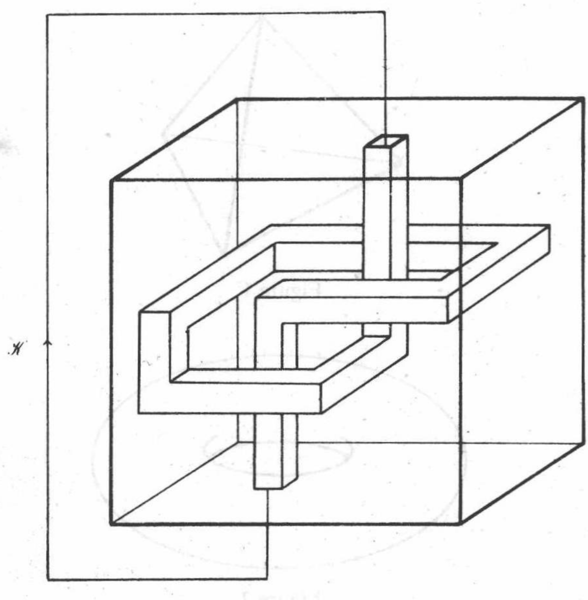


Figure 4

It seems very gratifying that the three dimensions provided by nature pose such a strong mathematical challenge. Moreover, it is known (Markov 1958) that the homeomorphism problem cannot be solved in dimensions ≥ 4, so we have every reason to concentrate our efforts in dimensions ≤ 3. This is the motivation for the present book. Our aim has been to give solutions to the main problems in dimension 2, and to select results in dimension 3 which illuminate the homeomorphism problem and seem likely to remain of interest if and when it is solved.

Like other fundamental problems in mathematics, the homeomorphism problem turns out not to be accessible directly, but requires various detours, some apparently technical and others of intrinsic interest. The first technical detour, which is typical, takes us away from the relation "is homeomorphic to" to the functions which relate homeomorphic figures. Thus we define a *homeomorphism* $f: \mathscr{A} \to \mathscr{B}$ to be a one-to-one continuous function with a continuous inverse $f^{-1}: \mathscr{B} \to \mathscr{A}$ (in particular, f is a bijection). Then to say $\mathscr{A}$ and $\mathscr{B}$ are homeomorphic is to say that there is a homeomorphism $f: \mathscr{A} \to \mathscr{B}$.

This point of view enables us to draw on general facts about continuous functions, which are reviewed in 0.1.2. We wish to avoid specific functions as far as possible, since topological properties by their nature do not reside in single functions so much as in classes of functions which are "qualitatively the same" in some sense. When we claim that there is a continuous function with particular qualitative features, it will always be straightforward to construct one by elementary means, such as piecing together finitely many linear functions. Readers should reassure themselves of this fact before proceeding too far, perhaps by working out explicit formulae for some of the examples in 0.1.3 (but not the "map of the Western Europe"!).

EXERCISE 0.1.1.1. Show that any two n-simplexes are homeomorphic.

EXERCISE 0.1.1.2. Construct a homeomorphism between the surface of a tetrahedron and the sphere.

0.1.2 Continuous Functions, Open and Closed Sets

The definition of a continuous function on $\mathbf{R}$, the real line, is probably familiar. We shall phrase this definition so that it applies to any space $\mathscr{S}$ for which there is a distance function $|P - Q|$ defined for all points P, Q. If $\mathscr{S} = \mathbf{R}^n$, which is the most general case we shall ultimately need, and if

$$P = (x_1, \ldots, x_n),$$
$$Q = (y_1, \ldots, y_n),$$

we have

$$|P - Q| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Then f is *continuous at P* if for each $\varepsilon > 0$ there is a δ such that

$$|P - Q| < \delta \quad \Rightarrow \quad |f(P) - f(Q)| < \varepsilon. \qquad (*)$$

The function f is simply called *continuous* if it is continuous at each point P in its domain.

Informally, we say that a continuous function sends neighbouring points to neighbouring points. In fact, if we define the *ε-neighbourhood* of a point X to be

$$\mathcal{N}_\varepsilon(X) = \{Y \in \mathcal{S} : |X - Y| < \varepsilon\},$$

then (*) says that any neighbourhood of $f(P)$ has all sufficiently small neighbourhoods of P mapped into it by f. (An ε-neighbourhood of a point is often called a *ball* neighbourhood because this is the actual form of the above set in the "typical" space R^3. One can generalize $\mathcal{N}_\varepsilon$ to any figure in an obvious way. We later consider ε-neighbourhoods of curves, which are "strips" in R^2 and "tubes" in R^3, and ε-neighbourhoods of surfaces, which are "plates.")

A set $\mathcal{O} \subset \mathcal{S}$ in which each point X has an $\mathcal{N}_\varepsilon(X) \subset \mathcal{O}$ is called *open* (in $\mathcal{S}$). Thus any space $\mathcal{S}$ is an open subset of itself, and the empty set $\varnothing$ is open for the silly reason that it has no elements to contradict the definition. More important examples are open intervals $\{x \in \mathsf{R} : a < x < b\}$ in the line R, and cartesian products of them in higher dimensions (rectangles in R^2, "hyperrectangles" in R^n).

The complement $\mathcal{C} = \mathcal{S} - \mathcal{O}$ of an open set $\mathcal{O}$ is called *closed* (in $\mathcal{S}$). The key property of a closed set is that it contains all its *limit points*. X is a limit point of a set $\mathcal{D}$ if every $\mathcal{N}_\varepsilon(X)$ contains a point of $\mathcal{D}$ other than X itself. It is immediate that a limit point X of $\mathcal{C}$ cannot lie in the open set $\mathcal{S} - \mathcal{C}$. If X is a limit point of both $\mathcal{D}$ and $\mathcal{S} - \mathcal{D}$ then X is called a *frontier point* of $\mathcal{D}$ and $\mathcal{S} - \mathcal{D}$, and the set of frontier points is called the *frontier* (of $\mathcal{D}$ and $\mathcal{S} - \mathcal{D}$). For example, the frontier of an n-simplex Δ^n in R^n is its boundary, while the frontier of a Δ^m in R^n, $m < n$, is Δ^m itself.

For every set $\mathcal{A}$ there is a smallest closed set $\bar{\mathcal{A}}$ containing it, and called its *closure*, and a largest open set $\text{int}(\mathcal{A})$ contained in it, and called its *interior*.

We now review some important properties of continuous functions, open sets, and closed sets.

(1) (Bolzano–Weierstrass theorem). *A closed set $\mathcal{C} \subset \mathsf{R}^n$ is bounded if and only if every infinite subset $\mathcal{D}$ of $\mathcal{C}$ has a limit point (in $\mathcal{C}$).*

If $\mathcal{C}$ is bounded, enclose it in a hyperrectangle and bisect repeatedly, each time choosing a half containing infinitely many points of $\mathcal{D}$. Doing this so that all edge lengths of the hyperrectangle $\to 0$ defines a point X which is a limit point of $\mathcal{D}$ by construction.

Conversely, if $\mathcal{C}$ is unbounded it contains a set $\mathcal{D} = \{P_i\}$ of points such that P_i is at distance ≥ 1 from $P_1, \ldots, P_{i-1}$ for each i, so $\mathcal{D}$ has no limit point. □

(2) *Two disjoint bounded closed sets $\mathscr{C}_1$, $\mathscr{C}_2$ have a non-zero distance $d(\mathscr{C}_1, \mathscr{C}_2)$ where*

$$d(\mathscr{C}_1, \mathscr{C}_2) = \inf\{|P_1 - P_2|: P_1 \in \mathscr{C}_1, P_2 \in \mathscr{C}_2\}$$

If $d(\mathscr{C}_1, \mathscr{C}_2) = 0$ choose $P_1^{(n)} \in \mathscr{C}_1$, $P_2^{(n)} \in \mathscr{C}_2$ for each n so that $|P_1^{(n)} - P_2^{(n)}| < 1/n$. If $\mathscr{C}_1$, $\mathscr{C}_2$ are disjoint this distance is always >0, hence the sets $\{P_1^{(i)}\}$ and $\{P_2^{(i)}\}$ are infinite and have limit points P_1, P_2 (by the Bolzano–Weierstrass Theorem) which are in $\mathscr{C}_1$, $\mathscr{C}_2$ respectively since the sets are closed. But then $|P_1 - P_2| > 0$, which contradicts the fact that P_1, P_2 are approached arbitrarily closely by $P_1^{(n)}, P_2^{(n)}$ which are arbitrarily close to each other. □

A bounded closed set in R^n is called *compact.* (By (1), an equivalent definition is that a compact set contains a limit point of each of its infinite subsets.) In many circumstances compact figures are equivalent to finite ones in the sense of 0.1.1, and this allows combinatorial arguments to be applied to rather general figures. Two propositions crucial to this "finitization" process are:

(3) *The continuous image of a compact set is compact.*

Let f be a function continuous on a compact set $\mathscr{C}$. By (1) it will suffice to show that every infinite $\mathscr{D} \subset f(\mathscr{C})$ has a limit point in $\mathscr{C}$. If not, there is an infinite set $\{f(X_i)\}$ of points in $f(\mathscr{C})$ with no limit point in $f(\mathscr{C})$. But $\{X_i\}$ has a limit point $X \in \mathscr{C}$ by (1), and every neighbourhood of $f(X)$ contains points $f(X_i)$ by the continuity of f, so $f(X)$ is a limit point of $\{f(X_i)\}$ and we have a contradiction. □

(4) *A continuous function f on a compact set $\mathscr{C} \subset \mathsf{R}^n$ is* uniformly continuous, *that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|X - Y| < \delta \Rightarrow |f(X) - f(Y)| < \varepsilon$$

regardless of the choice of $X, Y \in \mathscr{C}$.

Suppose on the contrary that there is no such δ for some fixed ε. Then there are $X_1, X_2, \ldots \in \mathscr{C}$ such that $\mathscr{N}_\delta(X_n)$ does not map into $\mathscr{N}_\varepsilon(f(X_n))$ unless $\delta < 1/n$. Let $X \in \mathscr{C}$ be a limit point of $\{X_1, X_2, \ldots\}$, using (1). Since f is continuous there is a $\delta > 0$ such that $\mathscr{N}_\delta(X)$ maps into $\mathscr{N}_{\varepsilon/2}(f(X))$.

Now for n sufficiently large we have not only $X_n \in \mathscr{N}_\delta(X)$, but also $\mathscr{N}_{1/n}(X_n) \subset \mathscr{N}_\delta(X)$, since X_n approaches arbitrarily close to X. Thus $\mathscr{N}_{1/n}(X_n)$ maps into $\mathscr{N}_{\varepsilon/2}(f(X))$, and in particular $f(X_n) \in \mathscr{N}_{\varepsilon/2}(f(X))$. But then $\mathscr{N}_{\varepsilon/2}(f(X)) \subset \mathscr{N}_\varepsilon(f(X_n))$ and hence $\mathscr{N}_{1/n}(X_n)$ maps into $\mathscr{N}_\varepsilon(f(X_n))$, contrary to the choice of X_n. □

For example, a curve c is a continuous map of the compact interval $[0, 1]$, so by (4) we can divide $[0, 1]$ into a finite number of subintervals (of

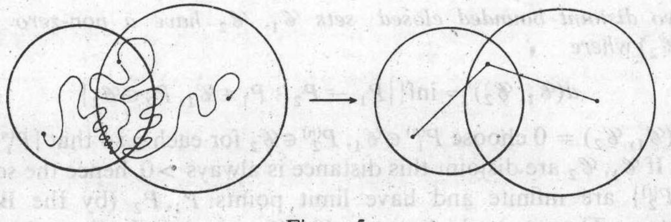


Figure 5

length $< \delta$) whose images (subarcs of c) lie in ε-neighbourhoods. If c lies in a figure with reasonable ε-neighbourhoods (say ε-balls, for ε sufficiently small), these subarcs can be deformed into line segments as in Figure 5. Thus c is equivalent to a polygonal curve, up to deformation. The notion of deformation required for this finitization process will be defined precisely in 0.1.9.

EXERCISE 0.1.2.1. If f is one-to-one consider the ordering of points on the curve $f(\mathscr{C})$ induced by the natural order on the line interval $\mathscr{C}$. Show that if $f(\mathscr{C})$ meets a closed set $\mathscr{K}$ then it has a *first* point of intersection with $\mathscr{K}$.

EXERCISE 0.1.2.2. The proofs of (1), (2), (3), (4) above use the Axiom of choice (where?). This can be avoided by giving an explicit rule for choosing a point $P(\mathscr{C})$ from a closed set $\mathscr{C} \subset \mathsf{R}^n$. Devise such a rule, starting in R^1.

EXERCISE 0.1.2.3. Construct a countable set of ball neighbourhoods in R^n, from which any open set is obtainable as the union of a subset. Deduce a rule for choosing a point from an open set.

EXERCISE 0.1.2.4. Show that a continuous one-to-one function on a bounded closed set has a continuous inverse (and hence is a homeomorphism).

EXERCISE 0.1.2.5. Show that an m-simplex is closed in any R^n, $n \geq m$.

EXERCISE 0.1.2.6. Show that $\overline{\mathscr{A}} = \mathscr{A} \cup \{\text{limit points of } \mathscr{A}\}$ and $\text{int}(\mathscr{A}) = \mathscr{S} - \overline{(\mathscr{S} - \mathscr{A})}$.

EXERCISE 0.1.2.7 (intermediate-value theorem). If $f: [a, b] \to \mathsf{R}$ is continuous, prove that f takes every value between $f(a)$ and $f(b)$.

0.1.3 Examples of Continuous Maps

Although it is superfluous to introduce another name for functions, we often call them *maps*, to emphasize the idea of a function as an image-forming process. This is particularly appropriate in topology, which owes its existence to the fact that some visual information is preserved even by arbitrary homeomorphisms. Homeomorphisms, or *topological maps*, can be called "maps" with some justice, and we extend the usage by courtesy to other continuous functions (though the continuous function which sends everything to the same point is a poor sort of "map"!).

Interestingly, modern geography has expanded its concept of "map" to virtually coincide with the general homeomorphism concept. One now sees maps in which each country is represented by a polygon, with area proportional not to its actual area, but to some other quantity such as population. The region being mapped nevertheless remains recognizable, mainly by the boundary relations between different countries, which are topologically invariant. Western Europe, for example, is shown in Figure 6.

However, we should not push the geographic analogy too far, as this can lead to the misconception that topology is just rubber sheet geometry, in other words, that all homeomorphisms are *deformations* (defined precisely as *isotopies* in 0.1.9). Once we leave the plane most of them are not—it is quite in order to cut a figure, deform it, and then rejoin, provided that rejoining restores the neighbourhood of each point on the cut. The torus provides a good illustration of this *cut and paste* method. In Figure 7 we cut the torus along a meridian a, twist one edge of the cut through 2π relative to the other, then rejoin. A small disc neighbourhood of any point on the cut is separated into semidiscs at the first step, but reunited after the twist of 2π, so for any ε-neighbourhood on the final torus we can find a δ-neighbourhood on the initial torus which maps into it. The transformation therefore defines a continuous one-to-one function, as does its inverse, so we have a homeomorphism f. It is intuitively clear that f cannot be realized by deformation alone, in particular b cannot be deformed onto $f(b)$. In fact, when one studies homeomorphisms of the torus algebraically (6.4) the deformations are factored out as trivial.

Continuous maps which are not necessarily one-to-one are also important. For example, a *curve* is nothing but a continuous map of a line segment. If

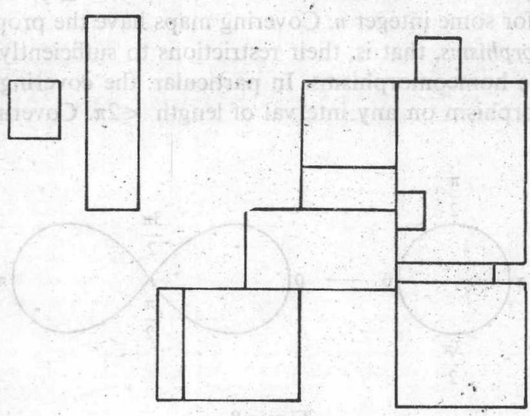


Figure 6