TEXTBOOKS in MATHEMATICS

A COURSE IN ABSTRACT HARMONIC ANALYSIS

Second Edition

Gerald B. Folland

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

TEXTBOOKS in MATHEMATICS

A COURSE IN ABSTRACT HARMONIC ANALYSIS

Second Edition

Gerald B. Folland

University of Washington
Seattle, Washington, USA

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group an Informa business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper
Version Date: 20150522

International Standard Book Number-13: 978-1-4987-2713-6 (Hardback)

Library of Congress Cataloging-in-Publication Data

Folland, G. B.
A course in abstract harmonic analysis / Gerald B. Folland. -- Second edition.
pages cm. -- (Textbooks in mathematics series ; 29)
"A CRC title."
Includes bibliographical references and index.
ISBN 978-1-4987-2713-6 (alk. paper)
1. Harmonic analysis. 2. Mathematical analysis. I. Title.

QA403.F58 2016
515'.2433--dc23 2015020328

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

A COURSE IN
ABSTRACT HARMONIC ANALYSIS
Second Edition

TEXTBOOKS in MATHEMATICS

Series Editors: Al Boggess and Ken Rosen

PUBLISHED TITLES

ABSTRACT ALGEBRA: AN INQUIRY-BASED APPROACH
Jonathan K. Hodge, Steven Schlicker, and Ted Sundstrom

ABSTRACT ALGEBRA: AN INTERACTIVE APPROACH
William Paulsen

ADVANCED CALCULUS: THEORY AND PRACTICE
John Srdjan Petrovic

ADVANCED LINEAR ALGEBRA
Nicholas Loehr

ADVANCED LINEAR ALGEBRA, SECOND EDITION
Bruce N. Cooperstein

ANALYSIS WITH ULTRASMALL NUMBERS
Karel Hrbacek, Olivier Lessmann, and Richard O'Donovan

APPLIED DIFFERENTIAL EQUATIONS: THE PRIMARY COURSE
Vladimir Dobrushkin

APPLYING ANALYTICS: A PRACTICAL APPROACH
Evan S. Levine

COMPUTATIONS OF IMPROPER REIMANN INTEGRALS
Ioannis Roussos

CONVEX ANALYSIS
Steven G. Krantz

COUNTEREXAMPLES: FROM ELEMENTARY CALCULUS TO THE BEGINNINGS OF ANALYSIS
Andrei Bourchtein and Ludmila Bourchtein

DIFFERENTIAL EQUATIONS: THEORY, TECHNIQUE, AND PRACTICE, SECOND EDITION
Steven G. Krantz

DIFFERENTIAL EQUATIONS WITH MATLAB®: EXPLORATION, APPLICATIONS, AND THEORY
Mark A. McKibben and Micah D. Webster

ELEMENTARY NUMBER THEORY
James S. Kraft and Lawrence C. Washington

PUBLISHED TITLES CONTINUED

ELEMENTS OF ADVANCED MATHEMATICS, THIRD EDITION
Steven G. Krantz

EXPLORING LINEAR ALGEBRA: LABS AND PROJECTS WITH MATHEMATICA®
Crista Arangala

AN INTRODUCTION TO NUMBER THEORY WITH CRYPTOGRAPHY
James Kraft and Larry Washington

AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS WITH MATLAB®, SECOND EDITION
Mathew Coleman

INTRODUCTION TO THE CALCULUS OF VARIATIONS AND CONTROL WITH MODERN APPLICATIONS
John T. Burns

INTRODUCTION TO MATHEMATICAL LOGIC, SIXTH EDITION
Elliott Mendelson

INTRODUCTION TO MATHEMATICAL PROOFS: A TRANSITION TO ADVANCED MATHEMATICS, SECOND EDITION
Charles E. Roberts, Jr.

LINEAR ALGEBRA, GEOMETRY AND TRANSFORMATION
Bruce Solomon

THE MATHEMATICS OF GAMES: AN INTRODUCTION TO PROBABILITY
David G. Taylor

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS, REVISED EDITION
Lawrence C. Evans and Ronald F. Gariepy

QUADRACTIC IRRATIONALS: AN INTRODUCTION TO CLASSICAL NUMBER THEORY
Franz Holter-Koch

REAL ANALYSIS AND FOUNDATIONS, THIRD EDITION
Steven G. Krantz

RISK ANALYSIS IN ENGINEERING AND ECONOMICS, SECOND EDITION
Bilal M. Ayyub

RISK MANAGEMENT AND SIMULATION
Aparna Gupta

TRANSFORMATIONAL PLANE GEOMETRY
Ronald N. Umble and Zhigang Han

Preface

Preface to the First Edition

The term "harmonic analysis" is a flexible one that has been used to denote a lot of different things. In this book I take it to mean those parts of analysis in which the action of a locally compact group plays an essential role: more specifically, the theory of unitary representations of locally compact groups, and the analysis of functions on such groups and their homogeneous spaces.

The purpose of this book is to give an exposition of the fundamental ideas and theorems of that portion of harmonic analysis that can be developed with minimal assumptions on the nature of the group with which one is working. This theory was mostly developed in the period from 1927 (the date of the Peter-Weyl theorem) through the 1960s. Since that time, research in harmonic analysis has proceeded in other directions, mostly on a more concrete level, so one may ask what is the excuse for a new book on the abstract theory at this time.

Well, in the first place, I submit that the material presented here is *beautiful.* I fell in love with it as a student, and this book is the fulfillment of a long-held promise to myself to return to it. In the second place, the abstract theory is still an indispensable foundation for the study of concrete cases; it shows what the general picture should look like and provides a number of results that are useful again and again. Moreover, the intervening years have produced few if any books with the scope of the present one. One can find expositions of various bits and pieces of this subject in a lot of places, and there are a few lengthy treatises in which one can perhaps learn more about certain aspects of it than one wants to know. But I have taken to heart the dictum propounded by R. F. Streater and A. S. Wightman in the preface of their book *PCT, Spin, Statistics, and All That*, that a book containing only Memorable Results is a Good Thing. The result, I hope, is a book that presents a rather large amount of important and interesting material in a concise and readable form.

The prerequisites for this book consist mostly of a familiarity with real analysis and elementary functional analysis. I use Folland [45] and Rudin [123] as standard references for this material; definitions and theorems in these books are used freely here, often without any specific reference. Rudin [123] also contains most of the material in Chapter 1, but the latter is included here because some of the concepts in it — especially projection-valued measures and the Gelfand transform — are an essential part of the fabric of ideas in later sections, and because I wished to include certain aspects of the spectral theorem that Rudin omits.

Chapters 2–6 are the core of the book. Chapter 2 develops the basic tools for doing analysis on groups and homogeneous spaces: invariant measures and the convolution product. Chapter 3 presents the rudiments of unitary representation theory, up through the Gelfand-Raikov existence theorem for irreducible unitary representations. In particular, it introduces the connection between representations and functions of positive type (or positive definite functions, as they are often called), an amazingly fruitful idea that also plays an important role in later chapters. Chapters 4 and 5 are devoted to analysis on Abelian and compact groups. Here the Fourier transform takes center stage, first as a straightforward generalization of the classical Fourier transform to locally compact Abelian groups, and then in the more representation-theoretic form that is appropriate to the non-Abelian case. Chapter 6 presents the theory of induced representations, including a complete proof of the Mackey imprimitivity theorem (something that is remarkably scarce in the expository literature) following the ideas of Blattner. In all these chapters, a number of specific examples are included to illustrate the general theory; they are interwoven with the rest of the text in Chapters 2–4 but are mostly collected in separate sections at the end in Chapters 5 and 6.

Chapter 7, on the theory of noncompact, non-Abelian groups, is of a somewhat different nature than the earlier chapters. To a considerable extent it is more like a survey article than a portion of a book, for many of the main results are stated without proof (but with references). To have given a complete treatment of the material in this chapter would have required the enlargement the book to an unwieldy size, involving a lengthy digression into the theory of von Neumann algebras and representations of C* algebras. (Indeed, many of the results are most naturally stated in this context, their application to groups coming via the group C* algebra.) The books of Dixmier [31], [32] already provide an excellent exposition of this theory, which I saw no reason to duplicate. Rather, I thought that many readers would appreciate a fairly detailed sketch of the Big Picture for noncompact, non-Abelian groups with the technical arguments omitted, especially since most of these results provide a background for the study of concrete cases rather than a set of working tools.

The Bibliography contains three kinds of items: original sources for the major results in the book, references for results stated without proof, and expository works to which readers can refer for more information on various topics. It makes no pretense of completeness. More extensive bibliographies can be found in Dixmier [32], Fell and Doran [40], [41], and Mackey [94], [98].

Chapters 2–5 are the embodiment of a course I gave at the University of Washington in the spring quarter of 1993. (The material of Chapter 1 was covered in a preceding course.) I wrote Chapters 6 and 7 while visiting the University of Colorado at Boulder for the fall semester of 1993, where I had the inestimable benefit of conversations with Arlan Ramsay and Larry Baggett. In addition, Baggett let me borrow some old handwritten notes by J. M. G. Fell, which were just what I needed to sort out many of the ideas in Chapter 6.

Many of the ideas in this book are an outgrowth of the study of the classical Fourier transform on the real line,

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\, dx.$$

Indeed, $\mathbb{R}$ is a locally compact group; the functions $e^{2\pi i x\xi}$ out of which $\mathcal{F}$ is fashioned are its irreducible representations; and $\mathcal{F}$ gives the Gelfand transform on $L^1(\mathbb{R})$, the spectral resolution of the algebra of translation-invariant operators on $L^2(\mathbb{R})$, and the decomposition of the regular representation of $\mathbb{R}$ into its irreducible components. When I first thought of writing a book like this, I envisaged it as an essay on the group-theoretic aspects of the Fourier transform. The scope of the book as it finally turned out is a bit different, but the spirit of Fourier is still all-pervasive.

Preface to the Second Edition

The main new features of this edition are as follows:

- In Chapter 1, I have added a short section (§1.6) on von Neumann algebras.
- In §4.5, I have included Mark Kac's simple proof of a restricted form of Wiener's theorem.
- I have rewritten part of §5.4 to explain the relation between $SU(2)$ and $SO(3)$ in terms of quaternions, an elegant method that also allows $SO(4)$ to be brought into the picture with little extra effort.
- In Chapter 6, I have added a discussion of the representations of the discrete Heisenberg group and its central quotients, which provides nice illustrations of both the Mackey machine for regular semi-direct

products and the pathological phenomena that occur for nonregular ones. In consequence, the "Examples" section has now expanded into two sections (§§6.7–8).

- I have moved the background material on tensor products of Hilbert spaces and operators from §7.3 to a new appendix.

In addition, I have added a few items to the "Notes and References" sections and a few entries to the Bibliography, clarified some obscurities, and corrected a number of typographical and mathematical errors. Of course, there may be some errors still remaining; as they are brought to my attention, they will be listed on an errata sheet linked to my home page: www.math.washington.edu/~folland/Homepage/index.html.

Some Matters of Notation and Terminology

The notation and terminology in this book agrees, for the most part, with that in Folland [45]. Here are a few specific items that are worthy of attention.

$\mathbb{T}$ denotes the multiplicative group of complex numbers of modulus one.

χ_E denotes the characteristic function or indicator function of the set E. If π is a finite-dimensional unitary representation, χ_π denotes its character. These two uses of the letter χ will cause no confusion.

In a topological space, a **neighborhood** of a point x or a set E is a set whose interior contains x or E. Thus, neighborhoods need not be open sets.

If X is a locally compact Hausdorff space, $C(X)$, $C_0(X)$, and $C_c(X)$ denote the spaces of continuous (complex-valued) functions on X, continuous functions vanishing at infinity, and continuous functions of compact support, respectively. (Of course, these spaces coincide when X is compact.) A **Radon measure** on X is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on open sets. (Outer and inner regularity on a set mean that the set can be approximated in measure from the outside or inside by open or compact sets, respectively. σ-finite Radon measures are **regular**, that is, both outer and inner regular on all Borel sets.)

The uniform norm is denoted by $\| \|_{\sup}$. (In [45] it is denoted by $\| \|_u$, but I found that this led to an unsightly overuse of the letter u in some situations.)

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the space of all bounded linear mappings from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, and the space of all bounded linear mappings from $\mathcal{X}$ to itself is denoted by $\mathcal{L}(\mathcal{X})$.

In §§2.2–4, left and right Haar measures on a locally compact group

G are denoted by λ and ρ. However, in §2.5 and for the remainder of the book, G is assumed to be equipped with a fixed left Haar measure, which is never given a name, and the symbols λ and ρ are freed for other purposes. The Haar measure of $E \subset G$ is denoted by $|E|$, the Lebesgue spaces of the Haar measure are denoted by $L^p(G)$ or simply L^p, and the Haar integral of $f \in L^1(G)$ is denoted by $\int f$ or $\int f(x)\,dx$.

Contents

1

Banach Algebras and Spectral Theory

This chapter contains a brief exposition of that part of Banach algebra theory that will be needed in the rest of this book, including the spectral theorem for commutative C* algebras. Although these topics are not part of harmonic analysis as such, the Gelfand transform and the spectral theorem are embodiments of ideas that are also central to harmonic analysis: the conversion of operators into more transparent forms and the decomposition of operators into simpler pieces.

1.1 Banach Algebras: Basic Concepts

A **Banach algebra** is an algebra $\mathcal{A}$ over the field of complex numbers equipped with a norm with respect to which it is a Banach space and which satisfies $\|xy\| \leq \|x\| \, \|y\|$ for all x, y in $\mathcal{A}$. $\mathcal{A}$ is called **unital** if it possesses a unit element or multiplicative identity, which we denote by e.

An **involution** on an algebra $\mathcal{A}$ is an anti-automorphism of $\mathcal{A}$ of order 2, that is, a map $x \mapsto x^*$ from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

$$(1.1) \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x$$

for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. An algebra equipped with an involution is called a **$*$-algebra**. A Banach $*$-algebra that satisfies

$$(1.2) \qquad \|x^* x\| = \|x\|^2 \text{ for all } x$$

is called a **C* algebra**.

We do not require an involution to satisfy $\|x^*\| = \|x\|$, although this holds for most of the examples we shall meet here. In particular it is true for C* algebras: the estimate $\|x\|^2 = \|x^* x\| \leq \|x^*\| \|x\|$ implies that $\|x\| \leq \|x^*\|$, and then $\|x^*\| \leq \|x^{**}\| = \|x\|$.

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, a (Banach algebra) **homomorphism** from $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are $*$-algebras, a **$*$-homomorphism** from $\mathcal{A}$ to $\mathcal{B}$ is a homomorphism ϕ such that $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{A}$.

If S is a subset of the Banach algebra $\mathcal{A}$, we say that $\mathcal{A}$ is **generated** by S if the linear combinations of products of elements of S are dense in $\mathcal{A}$.

We now describe four examples of Banach algebras. These examples barely begin to indicate how many different sorts of interesting Banach algebras there are, but they and their generalizations are the ones that will be important for us later.

Example 1. Let X be a compact Hausdorff space. The space $C(X)$ of continuous complex-valued functions on X is a unital Banach algebra with the usual pointwise algebra operations and the uniform norm. The map $f \mapsto \overline{f}$ is an involution that makes $C(X)$ into a C* algebra. Similarly, if X is a noncompact, locally compact Hausdorff space, $C_0(X)$ is a nonunital C* algebra.

If S is a set of functions in $C(X)$ (or $C_0(X)$) that separate points and have no common zeros, the Stone-Weierstrass theorem says that $C(X)$ (or $C_0(X)$) is generated by $S \cup \{\overline{f} : f \in S\}$.

Example 2. Let $\mathcal{H}$ be a Hilbert space. The set $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a unital Banach algebra, with the operator norm, and the map $T \mapsto T^*$ (T^* being the adjoint of T) is an involution that makes $\mathcal{L}(\mathcal{H})$ into a C* algebra. Here is the verification of (1.2): On the one hand, we have $\|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2$. On the other, for any unit vector $u \in \mathcal{H}$, $\|T^*T\| \ge \langle T^*Tu, u\rangle = \langle Tu, Tu\rangle = \|Tu\|^2$; taking the supremum over all such u we get $\|T^*T\| \ge \|T\|^2$. Any subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the operator norm and closed under taking adjoints is also a C* algebra.

Example 3. Let $l^1 = l^1(\mathbb{Z})$ be the space of all sequences $a = (a_n)_{-\infty}^{\infty}$ such that $\|a\| = \sum_{-\infty}^{\infty} |a_n| < \infty$. l^1 is a unital Banach algebra if we define multiplication to be convolution:

$$a * b = c, \text{ where } c_n = \sum_{-\infty}^{\infty} a_k b_{n-k}.$$

The unit element is δ, defined by $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$. The standard involution on l^1 is defined by

$$(a^*)_n = \overline{a}_{-n}.$$

l^1 is not a C* algebra with this involution; we leave it as an exercise for the reader to find a counterexample to (1.2).