Reductive Lie Groups

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Edited by **Michael Greco**

NY RESEARCH
PRESS
New York

Published by NY Research Press,
23 West, 55th Street, Suite 816,
New York, NY 10019, USA
www.nyresearchpress.com

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International Standard Book Number: 978-1-63238-399-0 (Hardback)

Printed in China.

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Preface

This book presents an extensive analysis of reductive lie groups. The aim of this profound book is to provide a comprehensive course on the topics of global study and establish certain orbital applications of the integration on topological groups and their algebras to harmonic analysis and induced representations in representation theory.

This book is a result of research of several months to collate the most relevant data in the field.

When I was approached with the idea of this book and the proposal to edit it, I was overwhelmed. It gave me an opportunity to reach out to all those who share a common interest with me in this field. I had 3 main parameters for editing this text:

1. Accuracy - The data and information provided in this book should be up-to-date and valuable to the readers.
2. Structure - The data must be presented in a structured format for easy understanding and better grasping of the readers.
3. Universal Approach - This book not only targets students but also experts and innovators in the field, thus my aim was to present topics which are of use to all.

Thus, it took me a couple of months to finish the editing of this book.

I would like to make a special mention of my publisher who considered me worthy of this opportunity and also supported me throughout the editing process. I would also like to thank the editing team at the back-end who extended their help whenever required.

Editor

Contents

Chapter 1

Introduction

I. 1. Introduction

In the study of the theory of *irreducible unitary representations,* is necessary to analyze and demonstrate diverse results on integral orbital of functions belonging to the cohomology $H^i(\mathfrak{g}, K; V \otimes V_\gamma^*)$, and that it is wanted they belong to the *$L^2(G)$-cohomology* of their reducible unitary representations called *discrete series.* Then is necessary consider the *Frèchet space* I(G), and analyze the *2-integrability* to the fibers of the space G/K, in spaces or locally compact components of G/K. For it will be useful the invariance of the corresponding measures of Haar under the actions of Ad(G), and the corresponding images of the *Harish-Chandra transform* on the space of functions $I_{a,b}(G)$.

Likewise, we will obtain a space in *cuspidal forms* that is an introspection of the class of the discrete series in the whole space G.

This *harmonic analysis* in the context of the space in cuspidal forms is useful in the exploration of the behavior of *characters* for those$(\mathfrak{g}, K)$-*modules* $H^i(\mathfrak{g}, K; V \otimes V_\gamma^*)$ and also for the generalization of the *integral formula of Plancherel* on *locally compact spaces* of G.

The generalization of the Plancherel formula is useful for the study of the functions on symmetrical spaces.

I. 2. Generalized spheres on Lie groups

We consider to $G = L/H_g$, a *homogeneous space* with origin $o = \{H_g\}$. Given $g_o \in G$, let L_{go} be the subgroup of G, letting g_o fix, that is to say; the subgroup of isotropy of G, in g_o.

Def. I. 2.1. A *generalized sphere* is an *orbit* $L_{go}g$, in G, of some point $g \in G$, under the subgroup of isotropy in some point $g_o \in G$.

In the case of a Lie group the generalized spheres are the left translations (or right) of their *conjugated classes.*

We assume that H_g, and each L_{go}, is *unimodular*. But is considering $L_{go}g = L_{go}/(L_{go})_g$, such that $(L_{go})_g$, be unimodular then the orbit $L_{go}g$, have an invariant measure determined except for a constant factor. Then are our interest the following general problems:

a. To determine a function f, on G, in terms of their orbital integrals on generalized spheres.

In this problem the essential part consist in the normalization of invariant measures on different orbits.

If is the case in that H_g, is compact, the problem A), is trivial, since each orbit $L_{g_0}g$, have finite invariant measure such that $f(g_0)$ is given as the limit when $g \to g_0$, of the variation of f, on $L_{g_0}g$.

I. 2.1. Orbits

Suppose that to every $g_0 \in G$, exist an open set L_{g0}-invariant $C_{g_0} \subset G$, containing g_0 in their classes such that to each $g \in C_{g0}$, the group of isotropy $(L_{g_0})_g$, is compact. The invariant measure on the orbit $L_{g0}g$ ($g_0 \in G$, $g \in C_{g_0}$) can be normalized consistently as follows: We fix a Haar measure dg_0, on L_0 ($H_g = L_0$). If $g_0 = g \mid o$, we have $L_{g0} = gL_0g^{-1}$,and we can to carry on dg_0, to the measure dg_{g0},on L_{g_0} through of the conjugation $z \to gzg^{-1}$ ($z \in L_0$). Since dg_0, is bi-invariant, $dg_{g_0,g}$, is independent of the election of g satisfying $g_0 = g \mid o$, the which is bi-invariant. Since $(L_{g_0})_g$, is compact, this have an only measure of Haar $dg_{g_0,g}$, with total measure 1 and reason why dg_0, and $dg_{g_0,g}$, determine canonically an invariant measure μ on the orbit$L_{g0}g = L_{g_0}/(L_{g_0})_g$.

Reason why also the following problem can to establish:

b. To express to $f(g_0)$, in terms of the integrals$\int_{L_{g0}g} f(p)d\mu(p)$, $g \in C_{g_0}$.

that is to say, the calculus of the orbital integrals on those measurable open sets called orbits.

I. 3. Invariant measures on homogeneous spaces

Let G, a locally compact topological group. Then a left invariant measure on G, is a positive measure, dg, on G, such that

$$\int_G f(xg)dg = \int_G f(g)\, dg, \tag{I. 3.1}$$

$\forall x \in G$, and all $f \in C_c(G)$. If G, is separable then is acquaintance (Haar theorem) that such measure exist and is unique except a multiplicative constant.

If G, is a Lie group with a finite number of components then a left invariant measure on G, can be identified with a left invariant n-form on G (where dim $G = n$). If μ, is a left invariant non-vanishing n-form on G, then the identification is implemented by the integration with regard to μ, using the canonical method of differential geometry. If G, is compact then we can (if is not specified) to use normalized left measures. This is those whose measure total is 1.

If dg is a left invariant measure and if $x \in G$, then we can define a new left invariant measure on G, μ_x, as follows:

$$\mu_x(f) = \int_G f(xg)dg, \qquad \text{(I. 3.2)}$$

The uniquely of the left invariant measure implies that

$$\mu_x(f) = \delta(x)\int_G f(xg)dg, \qquad \text{(I. 3.3)}$$

with δ, a function of x, which is usually called the modular function of G. If δ, is identically equal to 1,then we say that G, is unimodular. If G is then unimodular we can call to a left invariant measure (which is automatically right invariant) invariant. It is not difficult affirm that δ, is a continuous homomorphism of G, in the multiplicative group of positive real numbers. This implies that if G, is compact then G, is unimodular.

If G, is a Lie group, the modular function of G, is given by the following formula:

$$\delta(x) = |\det Ad(x)|, \qquad \text{(I. 3.4)}$$

where Ad, is the usual adjunct action of G, on their Lie algebra.

Let M, be a soft manifold and be μ, their form of volume. Let G, be a Lie group acting on M. Then $(g^*\mu)_x = c(g, x)\mu_x$, each $g \in G$, and $x \in M$. If is left as exercise verify that c satisfies the cocycle relationship

$$c(gh, x) = c(g, hx)c(h, x) \ \ \forall\ h, g \in G, x \in M, \qquad \text{(I. 3.5)}$$

We write as $\int_M f(x)dx$, to $\int_M f\mu$. The usual formula of change of variables implies that

$$\int_M f(gx)|c(g, x)|dx = \int_G f(x)dx, \qquad \text{(I. 3.6)}$$

to $f \in C_c(G)$, and $g \in G$.

Let H, be a closed subgroup of G. Be M = G/H. We assume that G, have a finite number of connect components. A G-invariant measure, dx, on M is a measure such that

$$\int_M f(gx)dx = \int_M f(x)dx, \ \ \forall\ f \in C_c(G), g \in G \qquad \text{(I. 3.7)}$$

If dx, comes of a form of volume on M, then (I. 3.7), is the same, which is equal to that $|c(g, x)| = 1\ \forall\ g \in G, x \in M$.

If M, is a soft manifold then is well acquaintance that M, have a form of volume or M, have a double covering that admit a form of volume. To rising of functions to the double covering (if it was necessary) one can integrate relatively to a form of volume on any manifold. Come back to the situation M = G/H, is not difficult demonstrate that M, admit a measure G-invariant if and only if the unimodular function of G, restricted to H, is equal to the unimodular function of H. Under this condition, a measure G-invariant on M is constructed as follow: Be $\mathfrak{g}$, the Lie algebra of G, and be $\mathfrak{h}$, the subalgebra of $\mathfrak{g}$, corresponding to H. Then we can to identify the tangent space of $1H$ to M with $\mathfrak{g}/\mathfrak{h}$. The adjunct action of H, on $\mathfrak{g}$, induces to action $\mathrm{Ad}^{\sim}$, of H, on $\mathfrak{g}/\mathfrak{h}$. The condition mentioned to obtain an identity in (I. 3. 7) tell us that $|\mathrm{Ad}^{\sim}(h)| = 1 \ \forall h \in H$. Thus if H^0, is the identity component of H, (as is usual) and if μ, is a element not vanishing of $\Lambda^m(\mathfrak{g}/\mathfrak{h})^*$[3] ($m = \dim G/H$) it is can to translate μ, to a form of G-invariant volume on G/H^0.

Therefore for rising of functions of M, to G/H^0, is had an invariant measure on M. But the Fubini theorem affirms that we can normalize dg, dh and dx, such that

$$\int_G f(g)dg = \int_{G/H} \left(\int_H f(gh)\, dh\right) d(gH), \quad f \in C_c(G) \qquad \text{(I. 3.8)}$$

Let G, be a Lie group with a finite number of connects components. Let H, be a closed subgroup of G, and let dh, be a selection of left invariant measure on H. The following result is used in the calculus of measures on homogeneous spaces.

Lemma I. 2.1. If f, is a compactly supported continuous function on H/G, (note the change to the right classes) then it exists, g, a continuous function supported compactly on such G, that

$$f(Hx) = \int_G g(hx)\, dh, \qquad \text{(I. 3.9)}$$

This result is usually demonstrated using a "partition of the unit" as principal argument.

For details of demonstration see [1].

Let G, be a Lie group and be A, and B, subgroups in G, such that A, and B, are compact and such that $G = AB$. The following result is used to the study of induced representations and classes of induced cohomology.

Lemma I. 2.2. We asume that G, is unimodular. If da, is a left invariant measure on A, and db, is a left invariant measure on B, then we can elect an invariant measure, dg, on G, such that

$$\int_G f(g)\, dg = \int_{A \times B} f(ab)\, da\, db, \quad \text{para } f \in C_c(G) \qquad \text{(I. 3.10)}$$

Proof: Consult [2]. ■

In the following section we will explain basic questions on invariant measures on homogeneous spaces. With it will stay clear the concept and use of normalized measures.

Let G, be a Lie group with Lie algebra $\mathfrak{g}$; let H, a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Each $x \in G$, gives rise to an analytic diffeomorphism

$$\tau(x) : gH \to xgH, \qquad \text{(I. 3.11)}$$

of G/H, onto itself. Let π, denote the natural mapping of G, onto G/H, and put $o = \pi(e)$. If $h \in H$, $(d\tau(h))_o$, is an endomorphism of the tangent space $(G/H)_o$. For simplicity, we shall write $d\tau(h)$, instead of $(d\tau(h))_o$, and $d\pi$, instead of $(d\pi)_e$ [4].

Lemma I. 2.3.

$$\det(d\tau(h)) = \frac{\det Ad_G(h)}{\det Ad_H(h)}, \qquad (I.\ 3.12)$$

$\forall h \in H$.

Proof. $d\pi$, is a linear mapping of $\mathfrak{g}$, onto $(G/H)_o$, and has kernel $\mathfrak{h}$. Let $\mathfrak{m}$, be any subspace of $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, (direct sum). Then $d\pi$, induces an isomorphism of $\mathfrak{m}$, onto $(G/H)_o$. Let $X \in \mathfrak{m}$. Then $Ad_G(h)X = dR_{h-1} \circ dL_h(X)$. Since $\pi \circ R_h = \pi$, $\forall\ h \in H$, and $\pi \circ L_g = \tau(g) \circ \pi$, $\forall\ g \in G$, we obtain

$$d\pi \circ Ad_G(h)X = d\tau(h) \circ d\pi(X), \qquad (I.\ 3.13)$$

The vector $Ad_G(h)X$, decomposes according to $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$,

$$Ad_G(h)X = X(h)_{\mathfrak{h}} + X(h)_{\mathfrak{m}}, \qquad (I.\ 3.14)$$

The endomorphism

$$A_h : X \to X(h)_{\mathfrak{m}}, \qquad (I.\ 3.15)$$

of $\mathfrak{m}$, satisfies

$$d\pi \circ Ad_h(X) = d\tau(h) \circ d\pi(X), \qquad (I.\ 3.16)$$

$\forall\ X \in \mathfrak{m}$, so $\det A_h = \det(d\tau(h))$. For other side,

$$exp\ Ad_G(h)tT = h\ exp\ tTh^{-1} = \exp exp\ Ad_H(h)tT, \qquad (I.\ 3.17)$$

for $t \in \mathbb{R}$, $T \in \mathfrak{h}$. Hence $Ad_G(h)T = Ad_H(h)T$, so

$$\det Ad_G(h) = \det A_h \det Ad_H(h), \qquad (I.\ 3.18)$$

and the lemma is proved. ■

Proposition I. 2.1. Let $m = \dim G/H$. The following conditions are equivalent:

i. G/H, has a nonzero G-invariant m-form ω;
ii. $\det Ad_G(h) = \det Ad_H(h)$, for $h \in H$.

If these conditions are satisfied, then G/H, has a G-invariant orientation and the G-invariant m-form ω, is unique up to a constant factor.

Proof. Let ω, be a G-invariant m-form on G/H, $\omega \neq 0$. Then the relation $\tau(h)^*\omega = \omega$, [3] at the point o, implies $\det(d\tau(h)) = 1$, so ii), holds. For other side, let $X_1, \ldots, X_m$, be a basis of $(G/H)_o$, and let $\omega^1, \ldots, \omega^m$, be the linear functions on $(G/H)_o$, determined by $\omega^i(X_j) = \delta_{ij}$. Consider the element$\omega^1\wedge\ldots\wedge\omega^m$, in the Grassmann algebra of the tangent space $(G/H)_o$. The condition ii), implies that $\det(d\tau(h)) = 1$, and the element $\omega^1\wedge\ldots\wedge\omega^m$, is invariant under the linear transformation $d\tau(h)$. It follows that exists a unique G-invariant m-form ω, on G/H, such that $\omega_o= \omega^1\wedge\ldots\wedge\omega^m$. If ω^*, is another G-invariant m-form on G/H, then $\omega^* = f\omega$, where $f\in C^\infty(G/H)$. Owing to the G-invariance, f = *constant*.

Assuming i), let $\varphi : p \rightarrow (x_1(p), \ldots, x_m(p))$, be a system of coordinates on an open connected neighborhood U, of $o\in G/H$, on which ω, has an expression

$$\omega_U = F(x_1,\ldots,x_m)\, dx_1\wedge\ldots\wedge dx_m,$$

With F > 0, The pair $(\tau(g)U, \varphi \circ \tau(g^{-1}))$, is a local chart on a connected neighborhood of $g \bullet o\in G/H$. We put $(\varphi \circ \tau(g^{-1}))(p) = (y_1(p), \ldots, y_m(p))$, for $p\in\tau(g)U$. Then the mapping

$$\tau(g) : U \rightarrow \tau(g)U,$$

has expression

$$(y_1, \ldots, y_m) = (x_1, \ldots, x_m).$$

On $\tau(g)U$, ω, has an expression

$$\omega_{\tau(g)U} = G(y_1,\ldots, y_m)\, dy_1\wedge\ldots\wedge dy_m,$$

and since $\omega_q= \tau(g)^*\omega_{\tau(g)q}$, we have for $q\in U \cap\tau(g)U$,

$$\omega_q= G(y_1(q),\ldots, y_m(q))\, (dy_1\wedge\ldots\wedge dy_m)_q= G(x_1(q), \ldots, x_m(q))\, (dx_1\wedge\ldots\wedge dx_m)_q,$$

Hence $F(x_1(q), \ldots, x_m(q)) = G(x_1(q), \ldots, x_m(q))$, and

$$F(x_1(q), \ldots, x_m(q)) = F(y_1(q),\ldots, y_m(q))[\partial(y_1(q),\ldots, y_m(q))/\partial(x_1(q), \ldots, x_m(q))],$$

which shows that the Jacobian of the mapping $(\varphi \circ \tau(g^{-1})) \circ \varphi^{-1}$, is positive. Consequently, the collection $(\tau(g)U, \varphi \circ \tau(g^{-1}))_{g\in G}$, of local charts turns G/H, into an oriented manifold and each $\tau(g)$, is orientation preserving. Then G-invariant form ω, now gives rise to an integral $\int f\omega$, which is invariant in the sense that

$$\int_{G/H} f\omega = \int_{G/H} (f \circ \tau(g))\omega, \quad \forall\ g\in G.$$

However, just as the Riemannian measure did not require orientability; an invariant measure can be construct on G/H, under a condition which is slightly more general than (ii). The projective$\mathbb{P}^2$, will, for example, satisfy this condition whereas it does not satisfy (ii). We recall that a measure μ, on G/H, is said to be invariant (or more precisely *G*-invariant) if $\mu(f \circ \tau(g)) = \mu(f)$, for all $g\in G$. ■

Theorem I. 2.1. Let G, be a Lie group and H, a closed subgroup. The following relation is satisfied

$$|det\ Ad_G(h)| = |det\ Ad_H(h)|, \quad h \in H, \qquad (I.\ 3.19)$$

Is a necessary and sufficient condition for the existence of a G-invariant positive measure on G/H. This measure dg_H, is unique (up to a constant factor) and

$$\int_G f(g)dg = \int_{G/H}\left(\int_H f(gh)dh\right)dg_H, \quad \forall\ f \in C_c(G), \qquad (I.\ 3.20)$$

if the left invariant measures dg, and dh, are suitably normalized.

Proof [6], [1]

Chapter 2

Integrals, Functional and Special Functions on Lie Groups and Lie Algebras

II. 1. Spherical functions

Let $P = P_{\Phi}$, be the minimal parabolic subgroup of G, with the Langlands decomposition

$$P = {}^{0}MAN, \qquad \text{(II. 1.1)}$$

If (σ, H^{σ}), is an irreducible unitary representation of ${}^{0}M$, and if $\mu \in (\mathfrak{a}_C)^*$, then $(\pi_{\sigma,\mu}, H^{\sigma,\mu})$, can denote the corresponding representation in principal serie. $H^{\sigma,\mu}$, is equivalent with $I_K(\sigma) = H^{\sigma}$. Indeed, if $H^{\sigma,\mu}$, is a representation of K, then exist $\sigma \in G$, such that $\sigma\mathfrak{a}^* = \mathfrak{a}^*$, to $\mathfrak{a}^*$, the dual algebra of the algebra $\mathfrak{a} \subset \mathfrak{p}$, since always there is a maximal Abelian subalgebra in $\mathfrak{p}$. Then $I_K(\sigma\mu) = I_K(\mu) = H^{\sigma,\mu}$. By the subcocient theorem to induced representations using the Casselman theorem, is possible to construct an operator that go from $Hom_{\mathfrak{a},K}(V, H^{\sigma})$, to $Hom_{\mathfrak{a},K}(V, H^{\sigma,\mu})$, that define an unitary equivalence between the representations in H^{σ}, and $H^{\sigma,\mu}$. Then H^{σ}, and $H^{\sigma,\mu}$, are equivalent representations as representations of the group K.

If $f \in H^{\sigma}$, that is to say, $f \in L^2({}^{0}M/K)$, then $f_{\mu}(nak) = a^{\mu+\rho}f(k)$, $\forall\, n \in N$, $a \in A$, and $k \in K$.

If $g \in G$, and $g = nak$, with $n \in N$, $a \in A$, and $k \in K$, then we can write $n(g) = n$, $a(g) = a$, $k(g) = k$. The theory of real reductive groups implies that as functions on G, n, a, and k, are smooth functions. We denote as **1**, to the function on K, that is identically equal to 1.

Let γ_0, be the class of the trivial representations of K. Then is clear that

$$(H^{\mu})_K(\gamma_0) = \mathbb{C}I, \qquad \text{(II. 1.2)}$$

If $\mu \in (\mathfrak{a}_C)^*$, then we define Ξ_{μ}, for

$$\Xi_{\mu}(g) = \langle \pi_{\mu}(g)\mathbf{1}_{\mu}, \mathbf{1}_{\mu} \rangle, \qquad \text{(II. 1.3)}$$

Said extended function to all the subgroup K, come given as

$$\Xi_{\mu}(g) = \int_K a(kg)^{\mu+\rho}dk, \ \ \forall\, g \in G \qquad \text{(II. 1.4)}$$

where $\mathbf{1}_{\mu}(g) = a(g)^{\mu+\rho}$, and $\mathbf{1}_{\mu}(k) = a(kg)^{\mu+\rho}$, $\forall\, g \in G$, $k \in K$.

Proposition II. 1.1. If $s \in W(\mathfrak{g}, \mathfrak{a})$, then $\Xi_{s\mu} = \Xi_{\mu}$, $\forall \mu \in (\mathfrak{a}_C)^*$.