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SYNTHESIS LECTURES ON COMPUTATIONAL ELECTROMAGNETICS

Constantine A. Balanis, *Series Editor*

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ABSTRACT

This lecture presents the perfectly matched layer (PML) absorbing boundary condition (ABC) used to simulate free space when solving the Maxwell equations with such finite methods as the finite difference time domain (FDTD) method or the finite element method. The frequency domain and the time domain equations are derived for the different forms of PML media, namely the split PML, the CPML, the NPML, and the uniaxial PML, in the cases of PMLs matched to isotropic, anisotropic, and dispersive media. The implementation of the PML ABC in the FDTD method is presented in detail. Propagation and reflection of waves in the discretized FDTD space are derived and discussed, with a special emphasis on the problem of evanescent waves. The optimization of the PML ABC is addressed in two typical applications of the FDTD method: first, wave-structure interaction problems, and secondly, waveguide problems. Finally, a review of the literature on the application of the PML ABC to other numerical techniques of electromagnetics and to other partial differential equations of physics is provided. In addition, a software package for computing the actual reflection from a FDTD-PML is provided. It is available at www.morganclaypool.com/page/berenger.

KEYWORDS

Absorbing boundary conditions, Perfectly matched layer, Numerical method, Finite difference, Finite element, Free space, Stretched coordinate, Discretized space, Evanescent wave, FDTD, PML

Contents

Introduction

Nowadays, computers have been used for several decades to solve the partial differential equations of physics. To this end, numerous computational methods have been developed. In the field of electromagnetics, some, such as the asymptotic methods, solve an approximation of the Maxwell equations. Others solve the exact Maxwell equations numerically, or a set equivalent to the Maxwell equations. The latter methods are the most widely used. They can be grouped into two classes: firstly the methods based on the solution of integral equations, secondly the finite methods that solve the Maxwell equations in a direct manner by discretizing the physical space with elementary volumes.

The integral equations have been extensively used since the 1960's. They permit realistic problems of practical interest to be solved with relatively modest computers. The most known integral method is the method of moments developed by Harrington [1] in frequency domain. The integral equations are equivalent to the Maxwell equations, the boundary conditions, and the initial conditions of the problem to be solved. They are solved on part of the physical space reduced to a surface or a region of space, depending on the problem. These numerical techniques do not require absorbing boundary conditions (ABCs) and will no longer be mentioned in the following.

Several finite methods have been developed for solving the Maxwell equations in a discretized space. The most popular is the finite-difference time-domain method (FDTD) introduced by K. S. Yee [2]. The finite volume method (FVTD), the transmission line matrix (TLM) method, and the finite element method (FEM) are finite methods as well. With all these numerical techniques the physical space is split into elementary cells, elements, or volumes, that must be smaller than both the shortest wavelength of interest and the smallest details of the geometry of the objects to be placed within the part of space of interest. Since the computers are not able, and will never be able, to handle an infinite number of elementary cells or elements, these methods only allow the Maxwell equations to be solved within a finite part of space. This is inconsistent with the requirements of most problems of electromagnetics that are unbounded problems. Consider for instance two typical problems of numerical electromagnetics, first the calculation of the radiation pattern of an antenna, second the interaction of an incident wave with a scattering structure. In both cases the radiated field propagates toward the free space surrounding the structure of interest; in other words the physical boundary conditions should be placed at infinity. If the Maxwell equations are solved within a finite volume bounded with arbitrary conditions, the solution is erroneous. In order to overcome such contradictory requirements, that is a physical unbounded space to be replaced with a finite computational domain, the so-called absorbing boundary conditions have been introduced.

The absorbing boundary conditions (ABCs) simulate or replace the infinite space that surrounds a finite computational domain. The replacement is never perfect. The solution computed within an ABC is only an estimate to the solution that would be computed within a really infinite domain. Moreover, the ABCs cannot replace sources of electromagnetic fields, they only absorb fields produced by sources located inside the surrounded domain. From this, sources cannot be placed outside the ABCs. As a corollary, the ABCs can be implemented only upon concave surfaces.

Various ABCs have been developed over the years in the field of electromagnetics, from the extrapolation [3] or the radiating boundary [4] in the 1970's to the perfectly matched layer (PML) [5] and the complementary operators method (COM) [6] in the 1990's. This lecture is devoted to the presentation of the PML ABC, initially introduced in [5] for use with the FDTD method. Since then, the PML ABC has been the subject of numerous works reported in the literature, with the objective of improving it, extending it to other numerical techniques of electromagnetics, and extending it to the solution of partial differential equations governing other domains of physics, such as acoustics, seismic, or hydrodynamics. The lecture is organized as follows:

- Chapter 1 discusses the requirements that must be fulfilled by the ABCs in view of replacing a theoretical infinite space with a finite computational domain. This chapter also reviews the ABCs that were used before the introduction of the PML ABC.
- Chapter 2 introduces the PML concept in the two-dimensional case.
- Chapter 3 extends the PML ABC to three dimensions and to general media. The PML medium is interpreted in terms of stretched coordinates and dependent currents, and the complex frequency shifted stretching factor is introduced.
- Chapter 4 derives the different forms of time domain equations, namely the split PML, the CPML, the NPML, the uniaxial PML, for a vacuum, lossy media, and more general anisotropic and dispersive media.
- Chapter 5 is devoted to the FDTD method. The FDTD equations are provided for the various forms of PML media. Propagation and reflection of waves in the discretized FDTD-PML space are derived theoretically and discussed, with a special emphasize on the case of evanescent waves.
- Chapter 6 presents the application of the PML ABC to two typical problems of numerical electromagnetics solved with the FDTD method, namely a wave-structure

interaction problem and a waveguide problem. The origin of spurious reflections from the PML is discussed and remedies are given so as to optimize the PML performance.

- Chapter 7 briefly reviews the extensions of the PML concept to other systems of coordinates, other numerical techniques, and other partial differential equations of physics.

CHAPTER 1

The Requirements for the Simulation of Free Space and a Review of Existing Absorbing Boundary Conditions

Answering two questions is the principal objective of this introductory chapter. The first question is: why is the simulation of free space needed in numerical electromagnetics? The second one is: which requirements have to be satisfied by the methods that simulate free space? In addition, the methods developed for simulating free space before the introduction of the perfectly matched layer concept are briefly reviewed.

1.1 THE MAXWELL EQUATIONS AND THE BOUNDARY CONDITIONS

The electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ in material media are governed by the Maxwell equations

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{1.1a}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \vec{J} \tag{1.1b}$$

with two Gauss laws satisfied at any time:

$$\nabla \cdot \mu \vec{H} = 0 \tag{1.2a}$$

$$\nabla \cdot \varepsilon \vec{E} = \rho . \tag{1.2b}$$

Permittivity ε and permeability μ are scalar quantities in isotropic media and tensor quantities in anisotropic media, $\boldsymbol{J}$ is a current density, and ρ is a charge density.

The Maxwell equations (1.1) are a set of two first-order partial differential equations connecting the time derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$ fields to some partial space derivatives of their

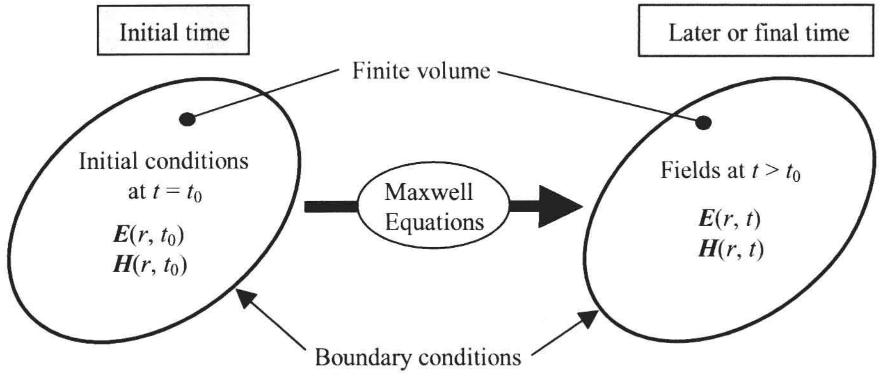

FIGURE 1.1: Evolution on time of the electromagnetic field governed by the Maxwell equations within a space domain bounded with boundary conditions

components. As known, this set of two equations can be merged into one second-order partial differential equation, namely the wave equation. As any partial differential equation or set of partial differential equations, the Maxwell equations are satisfied by an infinite number of solutions. In other words, there are an infinite number of physical problems that satisfy Eqs. (1.1). But there is only one solution that satisfies the following two additional conditions:

(1) *initial conditions*, that is $\boldsymbol{E}$ and $\boldsymbol{H}$ fields impressed within a given volume at an initial time,
(2) *boundary conditions*, that is $\boldsymbol{E}$ and $\boldsymbol{H}$ fields impressed at any time upon the whole surface enclosing the given volume.

The evolution in time of the initial $\boldsymbol{E}$ and $\boldsymbol{H}$ fields is governed by Eqs. (1.1) in conjunction with the boundary conditions. Initial $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are physical fields that satisfy (1.2). It is trivial to prove, by multiplying (1.1) with nabla operator, that the evolution in time preserves the satisfaction of (1.2). Solving a problem of electromagnetics, especially by means of numerical methods, consists of using the Maxwell equations (1.1) to advance in time the electromagnetic fields within a given part of space bounded with impressed boundary conditions, from an initial time to a later final time. This is summarized in Fig. 1.1. In principle, the finite methods are well suited to the solution of such problems. The volume of interest is discretized with a finite number of elementary volumes, called cells or elements, depending on the method. Nevertheless, an important difficulty arises as using finite methods, because in most applications the domain is, at least in theory, of infinite extent. This is discussed in the following.

1.2 THE ACTUAL PROBLEMS TO BE SOLVED WITH NUMERICAL METHODS

Ideally, a problem well suited to finite methods is like in Fig. 1.1, with a domain of resolution of the Maxwell equations as small as possible, limited to the region of interest, that is the region where the field has to be computed. This allows the number of elementary volumes and then the number of unknown fields to be as small as possible, or alternatively the discretization of space to be as fine as possible. Unfortunately, most problems encountered in numerical electromagnetics significantly differ from this ideal case. In most cases the domain of interest is not bounded with an impressed boundary condition. Instead, the region of interest is open, at least in part, to the surrounding free space. This means that the boundary condition is rejected to infinity, or equivalently that the computational domain is, in theory, infinite.

A popular problem involving an infinite domain is the calculation of the radiation pattern of an antenna. Only fields in the vicinity of the antenna are needed—the far fields can be obtained by a near-field to far-field transformation—but the antenna radiates in the surrounding free space. If an arbitrary boundary condition is placed at a finite distance from the antenna, the radiated field is reflected toward the inner domain, resulting in the addition of a spurious field to the solution in the vicinity of the antenna. In theory, this difficulty could be overcome with time domain methods, by working with a large domain in such a way that the fields reflected from the arbitrary boundary enter the region of interest after the end of the calculation. In actual applications, such a solution is not realistic, because the required computational domain would be so large that the problem could not be handled by the computers. From this, for the calculation of the field near an antenna with a finite method, the infinite space surrounding the antenna must be replaced with an appropriate boundary condition placed at a distance as short as possible from the antenna. This boundary condition must allow the fields computed in the domain to be a satisfactory approximation to the fields that would be obtained if the computational domain were infinite. Such a boundary condition is called an absorbing boundary condition (ABC) because it must remove the reflection of fields toward the inner domain, that is the ABC must absorb the radiated outgoing fields.

Problems that are close to antenna problems are the calculations of the interaction of an incident wave with a structure of interest. Such problems include radar cross-section (RCS) calculations and electromagnetic compatibility (EMC) calculations. The field scattered by the structure is radiated toward the surrounding infinite space. An ABC placed as close as possible to the structure is needed so as to replace the infinite free space and allow the overall domain to be as small as possible. This permits the computational resources to be devoted to the use of a discretization of the structure as fine as possible.

Besides problems open in totality to free space, there exist some problems that are only partially open. Examples can be found in the field of waveguides where most of the computational domain is bounded with the walls of waveguides. The domain is in general only open in one direction, for instance at one end of the waveguide. Nevertheless, an ABC is also needed in such partially open problems so as to limit to a reasonable size the computational domain.

1.3 THE REQUIREMENTS TO BE SATISFIED BY THE ABSORBING BOUNDARY CONDITIONS

Let us consider the field radiated from a small dipole antenna. In spherical coordinates (r, θ, φ), the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields are given by:

$$\vec{E}(r,\theta,\varphi) = \frac{-jIle^{-j\omega r/c}}{4\pi\varepsilon_0\omega}\left[2\cos\theta\left(\frac{1}{r^3}+\frac{j\omega}{cr^2}\right)\vec{u}_r + \sin\theta\left(\frac{1}{r^3}+\frac{j\omega}{cr^2}-\frac{\omega^2}{c^2r}\right)\vec{u}_\theta\right] \tag{1.3a}$$

$$\vec{H}(r,\theta,\varphi) = \frac{Ile^{-j\omega r/c}}{4\pi}\sin\theta\left(\frac{1}{r^2}+\frac{j\omega}{cr}\right)\vec{u}_\varphi \tag{1.3b}$$

where ω is the angular frequency, l is the dipole length, and I is the magnitude of the current upon the dipole. As known, far from the dipole ($r \gg$ wavelength), the radiated field (1.3) is like a homogeneous plane wave whose magnitude decreases as $1/r$. Conversely, at distances of the order of, or shorter than, the wavelength, the field is not homogeneous and its magnitude rapidly decreases with distance.

The behavior of the field radiated by a dipole is general. Far from any radiating or scattering structure the field is like a plane wave in a vacuum, with a magnitude decreasing as $1/r$. This is known as the Sommerfield radiation condition. Conversely, in the vicinity of the structure the field is not homogeneous and rapidly decreases with distance and its form is complex. Especially, this is the case around scattering structures stricken by an incident pulse. Strongly evanescent fields are present at frequencies lower than the resonance of the structure, up to a distance of the order of its size.

Other problems where evanescent fields are present near the domain of interest, are waveguide problems. Within a waveguide, both traveling and evanescent waves can exist. Below a cutoff angular frequency ω_{cutoff} the TE and TM modes are evanescent in the longitudinal direction of the waveguide. As an example, within a parallel-plate guide each mode is the superposition of two waves whose space dependence is of the form:

$$e^{-\eta j\frac{\omega}{c}\cosh\chi y}\, e^{\eta\frac{\omega}{c}\sinh\chi x} \tag{1.4a}$$

where x and y are the longitudinal and transverse directions, $\eta = \pm 1$, and:

$$\sinh \chi = \pm \sqrt{\frac{\omega_{\text{cutoff}}^2}{\omega^2} - 1}, \tag{1.4b}$$

with, for mode n and a guide of transverse size a:

$$\omega_{\text{cutoff}} = \frac{n\pi c}{a}. \tag{1.4c}$$

From this brief overview of the fields radiated or scattered in typical open problems of numerical electromagnetics, it appears that the requirements that an absorbing boundary condition must satisfy strongly depend on its location with respect to the source of the field:

- if the ABC is placed far from the source, the ABC only has to absorb homogeneous plane waves propagating with the speed of light c. In general the plane waves strike the boundary at oblique incidence.
- if the ABC is placed in the vicinity of the source, the ABC must be able to absorb nonhomogeneous evanescent waves. One might think that this requirement is more severe than only absorbing homogeneous traveling waves.

Equivalently, the above can be reformulated as follows:

- if the ABC is only able to absorb homogeneous plane waves, it must be placed out of the evanescent region surrounding the source (antenna, scattering structure, waveguide).
- if the ABC is able to absorb evanescent fields, it can be placed close to the source, in the evanescent region. In that case, the overall computational domain is significantly smaller.

1.4 THE EXISTING ABCs BEFORE THE INTRODUCTION OF THE PML ABC

From a general point of view, there exist two categories of absorbing boundary conditions:

- the global ABCs based on the fact that the field at any point on the boundary of a given volume can be expressed as a retarded-time integral of the field upon a surface enclosing all the sources [7]. Such global ABCs are computationally expensive and are only marginally used in numerical electromagnetics [8].
- the local ABCs with which the field on the boundary is expressed as a function of the field in the vicinity of the considered point, that is in function of the field at the closest points of the mesh with finite methods. All the ABCs used in the past in computer