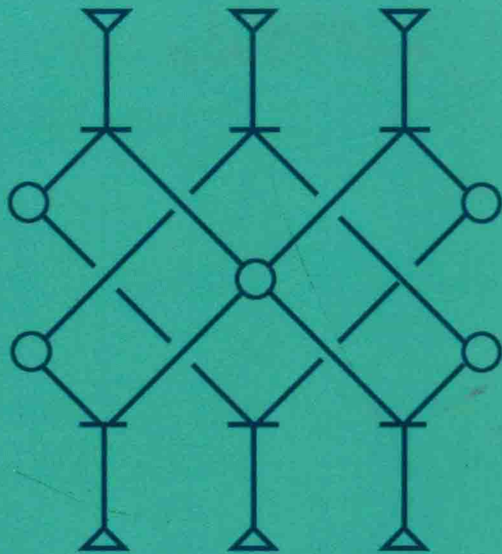

Multiple-Conclusion Logic

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Multiple-conclusion Logic

Preface

Logic is the science of argument, but ordinary arguments are lopsided: they can have any number of premisses but only one conclusion. In multiple-conclusion logic we allow any number of conclusions as well, regarding them (in Kneale's phrase) as setting out the field within which the truth must lie if the premisses are to be accepted. Thus we count a step in such an argument as valid if it is impossible for all its premisses to be true but all its conclusions false. Anything that can be said about premisses can now be said, mutatis mutandis, about conclusions. (For example, just as adding a member to a set of propositions makes more things follow from them, i.e. strengthens them as potential premisses, so it makes them follow from more things, i.e. weakens them as potential conclusions.) The subject owes much of its interest to the exploitation of this formal symmetry, while the contrasts between multiple- and single-conclusion calculi throw a fresh light on the conventional logic and its limitations.

Our subject is in its infancy. Its germ can be found in Gerhard Gentzen's celebrated *Untersuchungen über das logische Schliessen* (1934) if one is prepared to interpret his calculus of 'sequents' as a metatheory for a multiple-conclusion logic, but this is contrary to Gentzen's own interpretation, and it was Rudolf Carnap who first consciously broached the subject in his book *Formalization of logic* (1943). Carnap defined consequence and introduced rules of inference for multiple conclusions, but the first attempt to devise a proof technique to accommodate such rules was made by William Kneale in his paper *The province of logic* (1956).

Logicians in the subsequent decades appear to have ignored the lead of Carnap and Kneale. Doubtless this is because Kneale was mathematically an outsider and Carnap's jargon deterred even insiders; but doubtless too it reflects the prevalent conception of logic as the study of logical truth rather than logical consequence, and the property of theoremhood (deducibility from axioms) rather than the relation of deducibility in general. Such a climate is bound to be uncongenial to the development of multiple-conclusion logic, whose parity of treatment between premisses and conclusions calls for the more general approach from the outset. The reception given to Carnap's book by Church (1944) in his review is a notable case in point; but some recent Gentzen-inspired work on multiple and single cônclusions by Dana Scott, which we cite at the appropriate places in the text, is one of a number of welcome signs that things are changing.

Our book is in four parts, of which the second and third are independent of one another. The aim of Part I is to redefine the fundamental logical ideas so as to take account of multiple conclusions. We begin by recalling four methods of defining consequence for single-conclusion calculi, and abstract from them to produce four different though ultimately equivalent criteria for a relation to be a possible consequence relation. We then devise the appropriate multiple-conclusion analogues, paying special attention to the definition of proof and the somewhat complex sense in which consequence with multiple conclusions is transitive. For both kinds of calculus we discuss the theory of axiomatisability, which surprisingly is less straightforward for single-conclusion calculi than it is either for theories (sets of theorems) or for multiple-conclusion calculi; the idea of consequence by rules of inference, where this is defined in advance of any ideas of proof and indeed is used as a criterion for their adequacy; and rules with infinitely many premisses or conclusions.

The other main theme of Part I is the connection between multiple- and single-conclusion logic. It turns out that each multiple-conclusion calculus has a unique single-conclusion part, but each single-conclusion calculus has a range of multiple-conclusion counterparts. We investigate the composition of these ranges, the extent to which the properties of calculi of one kind can be predicted from their counterpart or counterparts of the other kind, and the connection between multiple conclusions and disjunction.

Part II starts from the observation that for an argument to be valid it is not enough that each of its component steps is valid in isolation: they must also relate to one another properly. In order to discuss this generally overlooked ingredient of validity we need a way of formalising arguments that displays their steps explicitly and unambiguously, and for this purpose we introduce the idea of an argument as a graph of formulae. This representation makes it possible to define the form of an argument ('form' being construed not with reference to any particular vocabulary of logical constants but as something shared by arguments of altogether different vocabularies) in purely syntactic terms. On the other hand our criterion of validity is ultimately semantic, involving the idea of consequence by rules introduced in Part I. We therefore investigate the connection between form and validity, looking for syntactic conditions for validity, trying to determine the adequacy or otherwise of various syntactically defined classes of proof, and discussing such topics as conciseness and relevance. We do this first for multiple-conclusion arguments (showing incidentally that Kneale's definition of proof is inadequate), then for single-conclusion ones and finally for arguments by infinite rules.

Our treatment of the subject so far is less concerned with particular calculi than with features of consequence and proof common to them all, but the remainder of the book introduces two specific applications. In Part III we make a detailed comparison between the multiple- and single-conclusion treatment of a particular topic. We choose many-valued logic as our example because it is well known and accessible, but there is also a historical reason. For it was Carnap's discovery of 'non-normal' interpretations of the classical propositional calculus - interpretations which fit the calculus but not the normal truth-tables - that led him to advocate multiple conclusions as the only fully satisfactory means of capturing truth-functional logic. We therefore include two-valued calculi in our discussion of many-valued ones, and investigate the multiple-conclusion counterparts of the classical calculus along with those of many-valued calculi in general. We show that every finite-valued multiple-conclusion propositional calculus is finitely axiomatisable and categorical (though virtually no single-conclusion calculus is categorical); and we pose the many-valuedness problem - the problem of distinguishing many-valued calculi from the rest by some intrinsic feature of their consequence relations. In general it seems that distinctions which were blurred in the single-conclusion case become sharp in the multiple-conclusion one, and results which needed qualification become unconditional. On the other hand it appears that a single-conclusion calculus displays a stability when its vocabulary is enlarged which a multiple-conclusion one may not; and one result of this is that we have had to leave the many-valuedness problem open for multiple-conclusion calculi though it is solved for single-conclusion ones.

In Part IV we explore the possibility of replacing the indirect methods of 'natural deduction' by direct proofs using multiple-conclusion rules, and with it the possibility of obtaining within our theory such results as the subformula theorem. We illustrate these ideas for the classical predicate calculus in a purely rule-theoretic context, and in a proof-theoretic one (presupposing some of the ideas of Part II) for the intuitionist propositional calculus.

Our results are published here for the first time apart from a short abstract (1973), but we have worked together on and off over six years and Chapter 19 has its origin in Shoesmith's Ph.D. dissertation (1962). We are much indebted to the trustees of the Radcliffe Trust for giving one of us a welcome relief from the pressures of other work by their award of a Radcliffe Fellowship to Smiley for 1970 and 1971.

Cambridge, 1976

We have taken the opportunity provided by a reprinting to make some minor corrections, and to remedy our misuse of the term 'recursive proof procedure' in Chapter 4, replacing it by 'recursive notion of proof'.

Cambridge, 1979

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PART II GRAPH PROOFS

PART III MANY-VALUED LOGIC

Introduction

A multiple-conclusion proof can have a number of conclusions, say $B_1,\dots,B_n$. It is not to be confused with a conventional proof whose conclusion is some one of the B_j, nor is it a bundle of conventional proofs having the various B_j for their respective conclusions: none of the B_j need be 'the' conclusion in the ordinary sense. This fact led Kneale to speak of the 'limits' of a 'development' of the premisses instead of the conclusions of a proof from them. We prefer to extend the sense of the existing terms, but hope to lessen one chance of misconstruction by speaking of a proof from $A_1,\dots,A_m$ *to* $B_1,\dots,B_n$ instead of a proof *of* $B_1,\dots,B_n$ from $A_1,\dots,A_m$.

The behaviour of multiple conclusions can best be understood by analogy with that of premisses. Premisses function collectively: a proof from $A_1,\dots,A_m$ is quite different from a bundle of proofs, one from A_1, another from A_2 and so on. Moreover they function together in a conjunctive way: to say that B follows from $A_1,\dots,A_m$ is to say that B must be true if A_1 and ... and A_m are true. Multiple conclusions also function collectively, but they do so in a disjunctive way: to say that $B_1,\dots,B_n$ follow from $A_1,\dots,A_m$ is to say that B_1 or ... or B_n must be true if all the A_i are true.

It should not be inferred from this explanation that multiple conclusions can simply be equated with the components of a single disjunctive conclusion $B_1 \vee \ldots \vee B_n$. The objections to such a facile reduction of multiple- to single-conclusion logic are the same as the objections to reducing the conventional logic to a logic of single premisses. It is true that any finite set of premisses is equivalent to a single conjunctive one, $A_1, \ldots, A_m$ having the same joint force as $A_1 \& \ldots \& A_m$. But this equivalence is only established by appealing to the workings of the rule 'from A,B infer A&B', understood as involving two separate premisses (not one conjunctive one), and it would be circular to appeal to the equivalence to establish the dispensability of the rule. Infinite or empty sets of premisses could not in any case be treated in this way; nor do all calculi possess a conjunction. Moreover, the equivalence between a set of sentences and their conjunction is at best a partial one, for the conjunction, being a sentence itself, can be made a component of further sentences where a set cannot: contrast $\sim(A_1 \& A_2)$ and $\sim\{A_1, A_2\}$. Our remark is that considerations exactly analogous to these apply to multiple conclusions and disjunctions.

To see how multiple conclusions might invite the attention of the logician, imagine first a student assigned the modest task of devising axioms and rules for the propositional calculus. He sees that the truth-table for conjunction can be translated immediately into rules of inference, the stipulation that A&B is true when A and B are true producing the rule (1) 'from A,B infer A&B', and similarly for (2) 'from A&B infer A' and (3) 'from A&B infer B'. Not only are these rules justified by the truth-table, but they in turn dictate it: any interpretation of conjunction that fits the rules must fit the truth-table too, for by (1) A&B must be true if, and by (2) and (3) only if, A and B are both true. Encouraged by this start the student moves on to disjunction, where the three 'true' cells in the truth-table immediately produce the rules ‘from A infer A∨B’ and ‘from B infer A∨B’. But when he comes to the remaining one - the entry ‘false’ when A and B are both false - the recipe fails. Moreover, even if he does find a complete set of rules, they cannot possibly dictate the intended interpretation of disjunction. For it is easy to show that all and only the tautologies and inferences of the propositional calculus are valid in the truth-tables below, where t stands for truth and f_1, f_2 and f_3 for subdivisions of falsity; yet A∨B can be true when A and B are both false. Our student has heard of the difficulties of excluding non-standard interpretations in the upper stories of mathematics; now he finds the same thing in the basement. He sees too that, if he could avail himself of it, the multiple-conclusion rule ‘from A∨B infer A,B’ would both translate the fourth cell of the original truth-table and serve to dictate the intended interpretation of disjunction in the same way as the rules for conjunction do.

&	t	f_1	f_2	f_3
t	t	f_1	f_2	f_3
f_1	f_1	f_1	f_3	f_3
f_2	f_2	f_3	f_2	f_3
f_3	f_3	f_3	f_3	f_3

∨	t	f_1	f_2	f_3
t	t	t	t	t
f_1	t	f_1	t	f_1
f_2	t	t	f_2	f_2
f_3	t	f_1	f_2	f_3

~	
t	f_3
f_1	f_2
f_2	f_1
f_3	t

Alternatively, consider the ambitious project of defining logic as advocated by Popper, Kneale and Hacking (for references see the historical note at the end of Section 2.1). It is proposed that a logical constant is one whose meaning can be explained by conventions governing its inferential behaviour. If the conventions are not merely to fix but to explain the meaning they must take the form of introduction rules, by which the behaviour of sentences containing the constant can be derived inductively from the behaviour of their constituents. (For example, using ⊢ to symbolise consequence and X for an arbitrary set of premisses, conjunction can be introduced by the rules 'if X ⊢ A and X ⊢ B then X ⊢ A&B' and 'if X,A ⊢ C or X,B ⊢ C then X,A&B ⊢ C'; but an elimination rule like 'if X ⊢ A&B then X ⊢ A' is ineligible, as is an appeal to the transitivity of ⊢.) Disjunction causes no difficulty this time, but material implication does. It turns out that the obvious introduction rules, 'if X,A ⊢ B then X ⊢ A⊃B' and 'if X ⊢ A and X,B ⊢ C then X,A⊃B ⊢ C', characterise intuitionist, not classical implication; and to introduce the latter it is necessary to have multiple conclusions. Indeed one would have to conclude that classical logicians, like so many Monsieur Jourdains, have been speaking multiple conclusions all their lives without knowing it.

No branch of mathematical logic relies exclusively on actual argumentative practice for its justification. We make use of an informal multiple-conclusion proof in Section 18.3, and note that the formalisation of a multiple-conclusion metacalculus provides a nice method of proving compactness (Theorem 13.1), but it can hardly be said that multiple-conclusion proofs form part of the everyday repertoire of mathematics. Perhaps the nearest one comes to them is in proof by cases, where one argues 'suppose A_1 ... then B, ... , suppose A_m ... then B; but $A_1 \vee \ldots \vee A_m$, so B'. A diagrammatic representation of this argument exhibits the downwards branching which we shall see is typical of formalised multiple-conclusion proofs:

$$\dfrac{A_1 \vee A_2 \vee \ldots \vee A_m}{\begin{array}{cccc} A_1 & A_2 & \ldots & A_m \\ \vdots & \vdots & & \vdots \\ B & B & & B \end{array}}$$