SAMIT KUMAR MAJUMDER

SOME TOPICS IN FUZZY ALGEBRA

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June, 2010 Samit Kumar Majumder

Thesis Dedicated

To My

Respected Parents

Contents

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

Chapter 1

INTRODUCTION AND PRELIMINARIES

In this chapter a brief out line of the present thesis has been given. Here we also recall some introductory concepts of fuzzy set theory, semigroup theory, ordered semigroup theory and Γ-semigroup theory.

1.1 Introduction

The concept of fuzzy sets was introduced by Lofti Zadeh[53] in his classic paper in 1965. Fuzzy sets are the further development of the mathematical concept of a set. Actually fuzzy set theory is a generalization of the classical set theory. Lofti Zadeh[53] defined fuzzy subset of a non-empty set as a collection of objects with grade of membership. After the introduction of fuzzy sets reconsideration of classical mathematics began. Now this theory has turned out to be a useful analytical device to analyze fruitfully those real world situations which are characterized by imprecision, vagueness and uncertainty. The concept of fuzzy set theory has been used by many researchers to generalize some of the basic notions of abstract algebra. As an immediate result fuzzy algebra is an well established branch of mathematics at present. Azirel Rosenfeld[42] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Rosenfeld[42] is the father of fuzzy abstract algebra. Nobuaki Kuroki[21, 22, 23] is the pioneer of fuzzy ideal theory of semigroups. Others who worked on fuzzy semigroup theory, such as X.Y. Xie[51, 52], Y.B. Jun[13, 14], are mentioned in the bibliography. In the last few years there appeared a good many papers on fuzzy semigroup theory and on intuitionistic fuzzy semigroup theory. In our present thesis we investigated some properties of fuzzy ideals in semigroups and partially ordered semigroups. We also unified the concepts of fuzzy translation and fuzzy multiplication of W.B. Vasantha Kandasamy[16] to introduce the idea of fuzzy magnified translation in semigroups and Γ-semigroups.

In 1981 M.K. Sen[44] introduced the notion of Γ-semigroup as a generalization of semigroup and ternary semigroup. We call this Γ-semigroup a *both sided Γ-semigroup*. In 1986 M.K. Sen and N.K. Saha[45] modified the definition of Sen's Γ-semigroup. We call here the newly defined Γ-semigroup a *one sided Γ-semigroup*. Γ-semigroups have been studied by a lot of mathematicians, for instance Chattopadhay[2, 3], T.K. Dutta and N.C. Adhikari[5, 6], Hila[10, 11], R. Chinram[4], Saha[43], Sen et al.[45, 46, 47], Seth[48]. Sen and Saha[43, 45, 46, 47] mostly worked on *one sided Γ-semigroups*. T.K. Dutta and N.C. Adhikari[5, 6] worked on *both sided Γ-semigroups*. They introduced the notion of operator semigroups of a Γ-semigroup. They established various relationships between ideals, prime ideals of a Γ-semigroup and that of its operator semigroups. Among other results they obtained inclusion preserving bijections between the set of all ideals, prime ideals of a Γ-semigroup and those of operator semigroups. Y.B. Jun[13] introduced the notion of fuzzy prime ideals of Γ-rings. T.K. Dutta and T. Chanda[8] studied fuzzy ideals, fuzzy prime ideals, fuzzy semiprime ideals of Γ-rings directly and via operator semigroups. In 2007, Uckun Mustafa, Ali Mehmet and Jun Young Bae[50] introduced the notions of intuitionistic fuzzy ideals in Γ-semigroups. In this thesis we introduce the concepts of fuzzy subsemigroups, fuzzy bi-ideals, fuzzy $(1,2)$-ideals, fuzzy quasi ideals, fuzzy ideals, fuzzy interior ideals, fuzzy prime and fuzzy semiprime ideals in a Γ-semigroup and study them via operator semigroups. We also obtain inclusion preserving bijections between the set of all fuzzy ideals, fuzzy prime and fuzzy semiprime ideals of a Γ-semigroup and those of its operator semigroups. These bijections are then used to give new proofs of its analogues in Γ-semigroups. We also introduce here the notion of fuzzy ideal extension of a Γ-semigroup and study them directly and via operator semigroups of a Γ-semigroup. It is important to note that in obtaining different useful results of Γ-semigroups in terms of fuzzy subsets the operator semigroups play equally effective role as it did outside the fuzzy setting.

The thesis consists of six chapters. **Chapter** 1, *i.e.*, the present Chapter is a brief introduction of the thesis. This also contains some preliminaries which we use in the sequel.

- In **Chapter 2**, the concept of fuzzy magnified translation in semigroups has been introduced. We have observed that fuzzy translation and fuzzy multiplication are the particular cases of fuzzy magnified translation. Here we have obtained some results on fuzzy ideal extensions in semigroups. Also we have verified different properties of fuzzy magnified translation under semigroup homomorphism. Among other results we obtain characterization of left regular, regular and intra-regular semigroups in terms of fuzzy magnified translation. All the works of this chapter have been published in **Advances in Fuzzy Sets and Systems**[29], **Bulletin of Pure and Applied Mathematics**[30] and **South East Asian Journal of Mathematics and Mathematical Sciences**[31].

- In **Chapter 3**, we have investigated some properties of fuzzy ideals in partially ordered semigroups. All the works of this chapter have been published in **Armenian Journal of Mathematics**[27] **.**

- In **Chapter 4**, the concepts of fuzzy subsemigroup, fuzzy bi-ideal, fuzzy $(1,2)$-ideal, fuzzy quasi ideal, fuzzy interior ideal and fuzzy ideal in a Γ-semigroup have been introduced. It is also observed here that they satisfy level subset criterion and characteristic function criterion. Here we obtain some results which are Γ-semigroup analogues of the results obtained by Kuroki in semigroups. In order to make operator semigroups of a Γ-semigroup work in the context of fuzzy sets as it worked in the study of Γ-semigroups, we obtain various relationships between fuzzy ideals of a Γ-semigroup and that of its operator semigroups. Here we obtain an inclusion preserving bijection between the set of all fuzzy ideals of a Γ-semigroup and that of its operator semigroups. This bijection is then used to give new proofs of its ideal analogue obtained in case of Γ-semigroups. Some of the works of this chapter have been published in **International Journal of Algebra**[32], **Computer and Mathematics with Applications**[35].

- **Chapter 5** is concerned with the introduction of the concept of fuzzy prime ideals and fuzzy semiprime ideals in Γ-semigroups. It is observed here that they satisfy level subset criterion and characteristic function criterion. Here we study fuzzy prime ideals and fuzzy semiprime ideals via operator semigroups also. We obtain inclusion preserving bijections between the set of all fuzzy prime(all fuzzy semiprime) ideals of a Γ-semigroup and that of its operator semigroups. Here we also introduce the notion of cartesian product of fuzzy prime and fuzzy semiprime ideals in a Γ-semigroup and study them directly and via operator semigroups of a Γ-semigroup. All the works of this chapter have been published in **International Journal of Pure and Applied Mathematics**[34], **International Journal of Contemporary Mathematical Sciences**[33].

- In **Chapter 6**, the concept of fuzzy ideal extensions in a semigroup introduced by X.Y.Xie[51] has been extended to the general situation of a Γ-semigroup. Here we obtain characterization of prime ideals of a Γ-semigroup in terms of fuzzy ideal extension. All the works of this chapter have been published in **International Mathematical Forum**[40], **International Journal of Contemporary Mathematical Sciences**[39].

1.2 Preliminaries

We recall some basic definitions and results of fuzzy set theory, semigroup theory, ordered semigroup theory and Γ-semigroup theory which will be required in the sequel.

Definition 1.2.1 [53] *A fuzzy subset of a non-empty set X is a function $\mu : X \to [0,1]$.*

Definition 1.2.2 [54]*A non-empty fuzzy subset $\mu : R^n \to [0,1]$ is called convex if $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \lambda\mu(x_1) + (1-\lambda)\mu(x_2)$, $\forall x_1, x_2 \in Supp(\mu)$, $\lambda \in [0,1]$.*

Definition 1.2.3 [9]*Let μ and σ be two fuzzy subsets of a set X. Then the cartesian product of μ and σ is defined by $(\mu \times \sigma)(x,y) = \min\{\mu(x), \sigma(y)\}$ $\forall x, y \in X$.*

Definition 1.2.4 [16]*Let μ be a fuzzy subset of a set X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. A mapping $\mu^T_\alpha : X \to [0,1]$ is called a fuzzy translation of μ if $\mu^T_\alpha(x) = \mu(x) + \alpha$ for all $x \in X$.*

Definition 1.2.5 [16]*Let μ be a fuzzy subset of a set X and $\beta \in [0,1]$. A mapping $\mu_\beta^M : X \to [0,1]$ is called a fuzzy multiplication of μ if $\mu_\beta^M(x) = \beta \cdot \mu(x)$ for all $x \in X$.*

Definition 1.2.6 [31]*Let μ be a fuzzy subset of a set X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}], \beta \in [0,1]$. A mapping $\mu_{\beta\alpha}^C : X \to [0,1]$ is called a fuzzy magnified translation of μ if $\mu_{\beta\alpha}^C(x) = \beta \cdot \mu(x) + \alpha$ for all $x \in X$.*

Definition 1.2.7 [21]*Let μ be a fuzzy subset of a set X. Then for $t \in [0,1]$ the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called t-level subset or simply level subset of μ.*

Definition 1.2.8 [41]*If $(X, *)$ is a mathematical system such that $\forall a, b, c \in X$, $(a * b) * c = a * (b * c)$, then $*$ is called associative and $(X, *)$ is called a semigroup.*

Definition 1.2.9 [41]*Let S be a semigroup. Let μ_1 and μ_2 be two fuzzy subsets of S. Then the product $\mu_1 \circ \mu_2$ of μ_1 and μ_2 is defined as*

$$(\mu_1 \circ \mu_2)(x) = \begin{cases} \sup\limits_{x=uv}[\min\{\mu_1(u), \mu_2(v)\} : u, v \in S] \\ 0, \text{ if for any } u, v \in S\ , x \neq uv \end{cases}$$

Definition 1.2.10 [22] *A non-empty subset A of a semigroup S is called a subsemigroup of S if $AA \subseteq A$.*

Definition 1.2.11 [22]*A non-empty subset A of a semigroup S is called a left(right) ideal of S if $SA \subseteq A$(resp. $AS \subseteq A$).*

Definition 1.2.12 [22]*A non-empty subset A of a semigroup S is called a two sided ideal(ideal) of S if it is both a left ideal and a right ideal of S.*

Definition 1.2.13 [14]*A subsemigroup A of a semigroup S is called an interior ideal of S if $SAS \subseteq A$.*

Definition 1.2.14 [4]*A non-empty subset A of a semigroup S is called a quasi ideal of S if $AS \cap SA \subseteq A$.*

Definition 1.2.15 [22]*A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$.*

Definition 1.2.16 [22]*A subsemigroup A of S is called an $(1,2)$-ideal of S if $ASAA \subseteq A$.*

Definition 1.2.17 [21]*A semigroup S is said to be left (right) regular if, for each element a of S, there exists an element x in S such that $a = xa^2$(resp. $a = a^2x$).*

Definition 1.2.18 [22]*A semigroup S is called intra-regular if for each element a of S, there exist elements $x, y \in S$ such that $a = xa^2y$.*

Definition 1.2.19 [22]*A semigroup S is called regular if for each element a of S, there exists an element $x \in S$ such that $a = axa$.*

Definition 1.2.20 [22]*A semigroup S is called archimedean if for all $a, b \in S$, there exists a positive integer n such that $a^n \in SbS$.*

Definition 1.2.21 [41]*Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as $AB = \{ab \in S : a \in A$ and $b \in B\}$.*

Theorem 1.2.1 [41]*A semigroup S is regular if and only if $R \cap L = RL$ for every right ideal R and every left ideal L of S.*

Definition 1.2.22 [41]*A non-empty fuzzy subset μ of a semigroup S is called a fuzzy subsemigroup of S if $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \quad \forall x, y \in S$.*

Definition 1.2.23 [41]*A fuzzy subsemigroup μ of a semigroup S is called a fuzzy bi-ideal of S if $\mu(xyz) \geq \min\{\mu(x), \mu(z)\} \quad \forall x, y, z \in S$.*

Definition 1.2.24 [21]*A fuzzy subsemigroup μ of a semigroup S is called a fuzzy $(1,2)$-ideal of S if $\mu(x\omega(yz)) \geqslant \min\{\mu(x), \mu(y), \mu(z)\} \; \forall x, \omega, y, z \in S$.*

Definition 1.2.25 [41]*A non-empty fuzzy subset μ of a semigroup S is called a fuzzy left(right) ideal of S if $\mu(xy) \geq \mu(y)$(resp. $\mu(xy) \geq \mu(x)$) $\forall x, y \in S$.*

Definition 1.2.26 [41]*A non-empty fuzzy subset μ of a semigroup S is called a fuzzy two-sided ideal or a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.*

Definition 1.2.27 [41]*A non-empty fuzzy subset* μ *of a semigroup* S *is called a fuzzy quasi-ideal of* S *if* $(\mu \circ \chi_S) \cap (\chi_S \circ \mu) \subseteq \mu$, *where* χ_S *is the characteristic function of* S.

Definition 1.2.28 [41]*A fuzzy subsemigroup* μ *of a semigroup* S *is called a fuzzy interior ideal of* S *if* $\mu(xay) \geq \mu(a)$ $\forall x, a, y \in S$.

Definition 1.2.29 [51]*A non-empty fuzzy subset* μ *of* S *is said to be a fuzzy prime ideal of* S *if* $\mu(xy) = \max\{\mu(x), \mu(y)\}$ $\forall x, y \in S$.

Definition 1.2.30 [21]*A non-empty fuzzy subset* μ *of a semigroup* S *is called a fuzzy semiprime ideal of* S *if* $\mu(x) \geq \mu(x^2)$ $\forall x \in S$.

Definition 1.2.31 [51]*Let* S *be a semigroup,* μ *a fuzzy subset of* S *and* $x \in S$. *The fuzzy subset* $< x, \mu >: S \to [0, 1]$ *is defined by* $< x, \mu > (y) = \mu(xy)$ *is called the extension of* μ *by* x.

Definition 1.2.32 [21]*Let* f *be a mapping from the set* X *to a set* Y. *If* λ *is a fuzzy subset of* Y, *then the preimage of* λ *under* f, *denoted by* $f^{-1}(\lambda)$ *is a fuzzy subset of* X *defined by* $f^{-1}(\lambda)(x) = \lambda(f(x))$ *for all* $x \in X$.

Lemma 1.2.1 [41]*Let* μ *be a fuzzy subset of a semigroup* S. *Then the following properties hold: (i)* μ *is a fuzzy subsemigroup of* S *if and only if* $\mu \circ \mu \subseteq \mu$, *(ii)* μ *is a fuzzy left(fuzzy right) ideal of* S *if and only if* $\chi_S \circ \mu \subseteq \mu$*(resp.* $\mu \circ \chi_S \subseteq \mu$*)* .

Definition 1.2.33 [25]*A po-semigroup(ordered semigroup) is an ordered set* $(S, ., \leq)$ *which is a semigroup such that for* $a, b \in S, a \leq b \Rightarrow xa \leq xb$ *and* $ax \leq bx$.

Definition 1.2.34 [25]*Let* S *be a po-semigroup. A non-empty subset* A *of* S *is said to be right(left) ideal of* S *if (i)* $AS \subseteq A$*(resp.* $SA \subseteq A$*), (ii)* $x \in A$ *and* $y \leq x$ *imply that* $y \in A$.

Definition 1.2.35 [25]*A non-empty subset* A *of a po-semigroup* S *is said to be an ideal if it is a right ideal and a left ideal of* S.

Definition 1.2.36 [25]*A po-semigroup* S *is called left(right) regular if for every* $a \in S$ *there exists* $x \in S$ *such that* $a \leq xa^2$ *(resp.* $a \leq a^2x$*).*