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Recent Progress in the Theory of the Euler and Navier–Stokes Equations

Edited by
James C. Robinson, José L. Rodrigo,
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To our families

Preface

This volume is the result of a workshop, "The Navier-Stokes Equations in Venice", which took place in the Palazzo Pesaro-Papafava in Venice (part of the *Warwick in Venice* program), April 8th–12th, 2013.

Several of the speakers agreed to write review papers related to their contributions to the workshop, while others have written more traditional research papers.We believe that this volume therefore provides an accessible summary of a wide range of active research topics, along with some exciting new results, and we hope that it will prove a useful resource for both graduate students new to the area and to more established researchers.

We would like to express their gratitude to the following sponsors of the workshop and the writing of this volume of proceedings: JCR was supported by an EPSRC Leadership Fellowship (grant EP/G007470/1). JLR is currently supported by the European Research Council (ERC grant agreement n. 616797).

Finally it is a pleasure to thank Chiara Croff, the Venice administrator of *Warwick in Venice*, for her assistance during the organization of the workshop.

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Classical solutions to the two-dimensional Euler equations and elliptic boundary value problems, an overview

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Abstract

Consider the classical initial, boundary-value problem for the 2D Euler equations, which describes the motion of an ideal, incompressible, fluid in a impermeable vessel. In the early eighties we introduced and studied a Banach space, denoted $C_*(\overline{\Omega})$, which enjoys the following property: if the curl of the initial velocity belongs to $C_*(\overline{\Omega})$, and the curl of the external forces is integrable in time with values in the above space $C_*(\overline{\Omega})$, then all derivatives appearing in the differential equations and in the boundary conditions are continuous in space-time, up to the boundary (we call these solutions *classical solutions*). At that time this conclusion was know if $C_*(\overline{\Omega})$ is replaced by a Hölder space $C^{0,\lambda}(\overline{\Omega})$. In the proof of the above result we appealed to a $C^2(\overline{\Omega})$ regularity result for solutions to the Poisson equation, vanishing on the boundary and with external forces in $C_*(\overline{\Omega})$. Actually, at that time, we have proved this regularity result for solutions to more general second-order linear elliptic boundary-value problems. However the proof remained unpublished. Recently, we have published an adaptation of the proof to solutions of the Stokes system. We recall these results in Section 1.1 below. On the other hand, attempts to prove the above regularity results for data in functional spaces properly containing $C_*(\overline{\Omega})$, have also been done. Bellow we prove some partial results in this direction. This possibly unfinished picture leads to interesting open problems.

Published in *Recent Progress in the Theory of the Euler and Navier-Stokes Equations*, edited by James C. Robinson, José L. Rodrigo, Witold Sadowski, & Alejandro Vidal-López.

1.1 The Euler and Stokes equations with data in $C_*(\overline{\Omega})$.

In these notes we want to give an overview on some results, both old and new. Some are old, but remained unpublished for a long time. The starting point will be Beirão da Veiga (1981, 1982, 1984).

We start by introducing some notation. Ω is an open, bounded, connected set in $\mathbb{R}^n$, $n \geq 2$, locally situated on one side of its boundary Γ. We assume that Γ is of class $C^{2,\lambda}(\overline{\Omega})$, for some positive λ. By $C(\overline{\Omega})$ we denote the Banach space of all real, continuous functions in $\overline{\Omega}$ with the norm

$$\|f\| \equiv \sup_{x\in\overline{\Omega}} |f(x)|.$$

In the sequel we use the notation

$$\|\nabla u\| = \sum_{i=1}^{n} \|\partial_i u\|, \quad \|\nabla^2 u\| = \sum_{i,j=1}^{n} \|\partial_{ij} u\|,$$

and appeal to the canonical spaces $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$, with the norms

$$\|u\|_1 \equiv \|u\| + \|\nabla u\|, \quad \|u\|_2 \equiv \|u\| + \|\nabla^2 u\|$$

respectively. Further, for each $\lambda \in (0, 1]$, we define the semi-norm

$$[f]_{0,\lambda} \equiv \sup_{x,y\in\Omega; x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\lambda}, \tag{1.1}$$

and the Hölder space $C^{0,\lambda}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_{0,\lambda} < \infty\}$, with the norm

$$\|f\|_{0,\lambda} = \|f\| + [f]_{0,\lambda}.$$

In particular, $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions in $\overline{\Omega}$. By $C^\infty(\overline{\Omega})$ we denote the set of all restrictions to $\overline{\Omega}$ of infinitely differentiable functions in $\mathbb{R}^n$. We will use boldface notation to denote vectors, vector spaces, and so on. We denote the components of a generic vector $\boldsymbol{u}$ by u_i, and similarly for tensors. Norms in functional spaces whose elements are vector fields are defined in the usual way, by appealing to the corresponding norms of the components.

In considering the two-dimensional Euler equations we will introduce the following well-known simplification. For a scalar function $u(x)$ (identified here with the third component of a vector field, normal to the plane of motion) we define the vector field $\text{Rot}u = (\partial_2 u, -\partial_1 u)$. For a vector field $\boldsymbol{v} = (v_1, v_2)$ we define the scalar field $\text{rot}\boldsymbol{v} = \partial_1 \boldsymbol{v}_2 - \partial_2 \boldsymbol{v}_1$ (the normal component of the curl). One has $-\Delta = \text{rotRot}$. Note that