Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Distributions and Nonlinear Partial Differential Equations

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to my wife HERMONA

P R E F A C E

The nonlinear method in the theory of distributions presented in this work is based on embeddings of the distributions in $\mathcal{D}'(R^n)$ into associative and commutative algebras whose elements are classes of sequences of smooth functions on R^n. The embeddings define various distribution multiplications. Positive powers can also be defined for certain distributions, as for instance the Dirac δ function.

A framework is in that way obtained for the study of nonlinear partial differential equations with weak or distribution solutions as well as for a whole range of irregular operations on distributions, encountered for instance in quantum mechanics.

In chapter 1, the general method of constructing the algebras containing the distributions and basic properties of these algebras are presented. The way the algebras are constructed can be interpreted as a sequential completion of the space of smooth functions on R^n. In chapter 2, based on an analysis of classes of singularities of piece wise smooth functions on R^n, situated on arbitrary closed subsets of R^n with smooth boundaries, for instance, locally finite families of smooth surfaces, the so called Dirac algebras, which prove to be useful in later applications are introduced.

Chapter 3 presents a first application. A general class of nonlinear partial differential equations, with polynomial nonlinearities is considered. These equations include among others, the nonlinear hyperbolic equations modelling the shock waves as well as well known second order nonlinear wave equations. It is shown that the piece wise smooth weak solutions of the general nonlinear equations considered, satisfy the equations in the usual algebraic sense, with the multiplication and derivatives in the algebras containing the distributions. It follows in particular that the same holds for the piece wise smooth shock wave solutions of nonlinear hyperbolic equations.

A second application is given in chapter 4, where one and three dimensional quantum particle motions in potentials arbitrary positive powers of the Dirac δ function are considered. These potentials which are no more measures, present the strongest local singularities studied in scattering theory. It is proved that the wave function solutions obtained within the algebras containing the distributions, possess the scattering property of being solutions of the potential free equations on either side of the potentials while satisfying special junction relations on the support of the potentials. In chapter 5, relations involving irregular products with Dirac distributions are proved to be valid within the algebras containing the distributions. In particular, several known relations in quantum mechanics, involving irregular products with

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Dirac and Heisenberg distributions are valid within the algebras. Chapter 6 presents the peculiar effect coordinate scaling has on Dirac distribution derivatives. That effect is a consequence of the condition of strong local presence the representations of the Dirac distribution satisfy in certain algebras. In chapter 7, local properties in the algebras are presented with the help of the notion of support, the local character of the product being one of the important results. Chapter 8 approaches the problem of vanishing and local vanishing of the sequences of smooth functions which generate the ideals used in the quotient construction giving the algebras containing the distributions. That problem proves to be closely connected with the necessary structure of the distribution multiplications. The method of sequential completion used in the construction of the algebras containing the distributions establishes a connection between the nonlinear theory of distributions presented in this work and the theory of algebras of continuous functions.

The present work resulted from an interest in the subject over the last few years and it was accomplished while the author was a member of the Applied Mathematics Group within the Department of Computer Science at Haifa Technion. In this respect, the author is particularly glad to express his special gratitude to Prof. A. Paz, the head of the department, for the kind support and understanding offered during the last years.

Many thanks go to the colleagues at Technion, M. Israeli and L. Shulman, for valuable reference indications, respectively for suggesting the scattering problem in potentials positive powers of the Dirac δ function, solved in chapter 4.

The author is indebted to Prof. B. Fuchssteiner from Paderborn, for his suggestions in contacting persons with the same research interest.

Lately, the author has learnt about a series of extensive papers of K. Keller, from the Institute for Theoretical Physics at Aachen, presenting a rather complementary approach to the problem of irregular operations with distributions. The author is very glad to thank him for the kind and thorough exchange of views.

A special gratitude and acknowledgement is expressed by the author to R.C. King from Southampton University, for his generosity in promptly offering the result on generalized Vandermonde determinants which corrects an earlier conjecture of the author and upon which the chapters 5 and 6 are based.

All the highly careful and demanding work of editing the manuscript was done by my wife Hermona, who inspite and on the account of her other much more interesting and elevated usual occupations found it necessary to support an effort in regularizing

the irregulars ..., in multiplying the distributions ...

By the way of multiplication: Prof. A. Ben-Israel, a former colleague, noticing the series of preprints, papers, etc. resulted from the author's interest in the subject and seemingly inspired by one of the basic commandments in the Bible, once quipped: "Be fruitful and multiply ... distributions ..."

E. E. R.

Haifa, December 1977

CONTENT

NOTE

The Reader interested mainly in NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS, may at a first lecture concentrate on the following sections:

"Never forget

the beaches of ASHQELON ... "

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C h a p t e r 1

ASSOCIATIVE, COMMUTATIVE ALGEBRAS CONTAINING THE DISTRIBUTIONS

§1. NONLINEAR PROBLEMS

The theory of distributions has proved to be essential in the study of linear partial differential equations. The general results concerning the existence of elementary solutions, [103], [34], P-convexity as the necessary and sufficient condition for the existence of smooth solutions, [103], the algebraic characterization of hypoellipticity, [64], etc., are several of the achievements due to the distributional approach, [154], [63], [64], [153], [156], [33], [114].

In the case of nonlinear partial differential equations certain facts have pointed out the useful role a nonlinear theory of distributions could play. For instance, the appearance of shock discontinuities in the solutions of nonlinear hyperbolic partial differential equations, even in the case of analytic initial data, [62], [89], [113], [50] [70], [51], [24], [25], [26], [31], [32], [52], [58], [71], [79], [84], [90], [91], [133], [149], [163], indicates that in the nonlinear case problems arise starting with a rigorous and general definition of the notion of solution. Important cases of nonlinear wave equations, [5], [9], [10], [11], [121], prove to possess distribution solutions of physical interest, provided that 'irregular' operations, e.g. products, with distributions are defined. Using suitable procedures, distribution solutions can be associated to various nonlinear differential or partial differential equations, [1], [2] [30], [42], [43], [45], [80], [92], [94], [117], [118], [119], [120], [122], [138], [146], [147], [159], [160], [161].

In quantum mechanics, procedures of regularizing divergent expressions containing 'irregular' operations with distributions, such as products, powers, convolutions, etc., have been in use, [6], [7], [12], [13], [15], [19], [20], [21], [29], [54], [55], [56] [57], [60], [76], [77], [78], [112], [143], [151], suggesting the utility of enriching in a systematic way the vector space structure of the distributions.

A natural way to start a nonlinear theory of distributions is by supplementing the vector space structure of $\mathcal{D}'(R^n)$ with a suitable distribution multiplication.

Within this work, a nonlinear method in the theory of distributions is presented, based on an associative and commutative multiplication defined for the distributions in $\mathcal{D}'(R^n)$, [125-131]. That multiplication offers the possibility of defining arbitrary positive powers for certain distributions, e.g. the Dirac δ function, [130], [151].

The definition of the multiplication rests upon an analysis of classes of singularities of piece wise smooth functions on R^n, situated on arbitrary closed subsets of R^n with smooth boundaries, for instance locally finite families of smooth surfaces in R^n (chap. 2, § 3).

Several applications are presented.
First, in chapter 3, it is shown that the piece wise smooth weak solutions of a general class of nonlinear partial differential equations satisfy those equations in the usual algebraic sense, with the multiplication and derivatives in the algebras containing the distributions. As a particular case, it results that the piece wise smooth nonlinear shock wave solutions of the equation, [90], [71], [133], [52], [32] [131]:

$$u_t(x,t) + a(u(x,t)) \cdot u_x(x,t) = 0 \ , \ x \in R^1 \ , \ t > 0 \ ,$$
$$u(x,0) = u_o(x) \ , \ x \in R^1 \ ,$$

where a is an arbitrary polynomial in u, satisfy that equation in the usual algebraic sense.

Second, in chapter 4, quantum particle motions in potentials arbitrary positive powers of the Dirac δ distribution are considered. These potentials present the strongest local singularities studied in recent literature on scattering, [27], [3], [28], [115], [116], [140]. The one dimensional motion has the wave function ψ given by

$$\psi''(x) + (k-\alpha(\delta(x))^m)\psi(x) = 0 \ , \ x \in R^1 \ , \ k,\alpha \in R^1 \ , \ m \in (0,\infty]$$

while the three dimensional motion assumed spherically symmetric and with zero angular momentum has the radial wave function R given by

$$(r^2R'(r))' + r^2(k-\alpha(\delta(r-a))^m)R(r) = 0 \ , \ r \in (0,\infty) \ , \ k,\alpha \in R^1 \ , \ a,m \in (0,\infty) \ .$$

The wave function solutions obtained possess a usual scattering property, namely they consist of pairs ψ_-,ψ_+ of usual C^∞ solutions of the potential free equations, each valid on the respective side of the potential while satisfying special junction relations on the support of the potentials.

Third, it is shown in chap. 5, §5, that the following well known relations in quantum mechanics, [108], involving the square of the Dirac δ and Heisenberg δ_+,δ_- distributions and other irregular products hold:

$$(\delta)^2 - (1/x)^2/\pi^2 = -(1/x^2)/\pi^2$$

$$\delta \cdot (1/x) = -D\delta/2$$

$$(\delta_+)^2 = -D\delta/4\pi i - (1/x^2)/4\pi^2$$

$$(\delta_-)^2 = D\delta/4\pi i - (1/x^2)/4\pi^2$$

where $\delta_+ = (\delta+(1/x)/\pi i)/2$, $\delta_- = (\delta-(1/x)/\pi i)/2$.

§2. MOTIVATION OF THE APPROACH

The distribution multiplication, defined for any given pair of distributions in $\mathcal{D}'(R^n)$, could either lead again to a distribution or to a more general entity. Taking into account H. Lewy's simple example, [93] (see also [64], [155], [48]), of a first order linear partial differential operator with three independent variables and coefficients polynomials of degree at most one with no distribution solutions, the choice of a distribution multiplication which could in the case of particularly irregular factors lead outside of the distributions, seems worthwhile considering. Such an extension beyond the distributions would mean an increase in the 'reservoir' of both data and possible solutions of nonlinear partial differential operators, not unlike it happened with the introduction of distributions in the study of linear partial differential operators, [154].

One can obtain a distribution multiplication in line with the above remarks by embedding $\mathcal{D}'(R^n)$ into an algebra A *). It would be desirable for a usual Calculus if the algebra A were associative, commutative, with the function $\psi(x) = 1$, $\forall\, x \in R^n$, its unit element and possessing derivative operators satisfying Leibnitz type rules for the product derivatives. Certain supplementary properties of the embedding $\mathcal{D}'(R^n) \subset A$ concerning multiplication, derivative, etc. could also be envisioned.

There is a particularly convenient classical way to obtain such an algebra A , namely, as a sequential completion of $\mathcal{D}'(R^n)$ or eventually, of a subspace F in $\mathcal{D}'(R^n)$. The sequential completion, suggested by Cauchy and Bolzano, [158], was employed rigorously by Cantor, [22], in the construction of R^1 . Within the theory of distributions the sequential completion was first employed by J. Mikusinski, [105] (see also [110]) in order to construct the distributions in $\mathcal{D}'(R^1)$ from the set of locally integrable functions on R^1 , however without aiming at defining a distribution multiplication.

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*) All the algebras in the sequel are considered over the field C^1 of the complex numbers.

Later, in [106], the problem of a whole range of 'irregular' operations - among them, multiplication - was formulated within the framework of the sequential completion.

The method of the sequential completion possesses two important advantages. First, there exist various subspaces F in $\mathcal{D}'(R^n)$ which are in a natural way associative, commutative algebras, with the unit element the function $\psi(x) = 1$, $\forall x \in R^n$. Starting with such a subspace F, it is easy to construct a sequential completion A which will also be an associative, commutative algebra with unit element. Indeed, the procedure is - from purely algebraic point of view - the following one. Denote $\mathcal{W} = N \to F$, that is, the set of all sequences with elements in F. With the term by term operations on sequences, $\mathcal{W}$ is an associative, commutative algebra with unit element. Choosing a suitable subalgebra $\mathcal{A}$ in $\mathcal{W}$ and an ideal in $\mathcal{A}$, one obtains $A = \mathcal{A}/\mathcal{I}$.

Second, the sequential completion A results in a constructive way. Further, a simple characterization of the elements in A is obtained. Indeed, these elements will be classes of sequences of 'regular' functions in R^n (in this work, $F = \mathcal{C}^\infty(R^n)$ will be considered) much in the spirit of various 'weak solutions' used in the study of partial differential equations.

Within the more general framework of Calculus, the distributional approach - essentially a sequential completion of a function space, [105],[110],[4] - can be viewed as a stage in a succession of attempts to define the notion of function. Euler's idea of function, as an analytic one was extended by Dirichlet's definition accepting any univalent correspondence from numbers to numbers. That extension although significant - encompassing even nonmeasurable functions, provided the Axiom of Choice is assumed, [49] - failed to include certain rather simple important cases, as for instance, the Dirac δ function and its derivatives.

It is worthwhile mentioning that the distributional approach can be paralleled by certain approaches in Nonstandard Analysis. In [134], a nonstandard model of R^1 obtained by a sequential completion of the rational numbers was presented. In that nonstandard R^1, the Dirac δ function becomes a usual univalent correspondence from numbers (nonstandard) to numbers (nonstandard).

The notion of the germ of a function at a point which can be regarded as a generalization of the notion of function, since it represents more than the value of the function at the point but less than the function on any given neighbourhood of the point, is related both to the distributional approach and Nonstandard Analysis, [109],[97].

The variety of interrelated approaches suggests that the notion of function in Calculus is still 'in the making'. The particular success of the distributional approach in the theory of linear partial differential equations (especially the constant coefficient case, otherwise see [93]) is in a good deal traceable to the strong results and methods in linear functional analysis and functions of several complex variables. In this respect, the distributional approach of nonlinear problems, such as nonlinear partial differential equations, can be seen as requiring a return to more basic and general methods, as for instance, the sequential completion of convenient function spaces, which finds a natural framework in the theory of Algebras of Continuous Functions (see chap. 8).

The sequential completion is a common method for both standard and nonstandard methods in Calculus and its theoretical importance is supplemented by the fact that it synthetizes basic approximation methods used in applications, such as the method of 'weak solutions'. The nonlinear method in the theory of distributions presented in this work is based on the embedding of $\mathcal{D}'(R^n)$ into associative and commutative algebras with unit element, constructed by particular sequential completions of $\mathcal{C}^\infty(R^n)$, resulting from an analysis of classes of singularities of piece wise smooth functions on R^n, situated on arbitrary closed subsets of R^n with smooth boundaries, for instance locally finite families of smooth surfaces in R^n (see chap. 2, §3).

§3. DISTRIBUTION MULTIPLICATION

The problem of distribution multiplication appeared early in the theory of distributions, [135], [81-83], and generated a literature, [9], [11-21], [35-41], [46], [53-57], [61], [66-69], [72-74], [76-78], [85], [106-108], [112], [125-132], [134], [136] [137], [148], [151], [162]. L. Schwartz's paper [135], presented a first account of the difficulties. Namely, it was shown impossible to embed $\mathcal{D}'(R^1)$ into an associative algebra A under the following conditions:

a) the function $\psi(x) = 1$, $\forall\, x \in R^1$, is the unit element of the algebra A ;

b) the multiplication in A of any two of the functions

$$1, x, x(\ell n|x|-1) \in \mathcal{C}^0(R^1)$$

is identical with the usual multiplication in $\mathcal{C}^0(R^1)$;

c) there exists a linear mapping (generalized derivative operator) $D : A \to A$, such that:

c.1) D satisfies on A the Leibnitz rule of product derivative:

$$D(a\cdot b) = (Da)\cdot b + a\cdot(Db) \ , \quad \forall\, a,b \in A \ ;$$

c.2) D applied to the functions

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