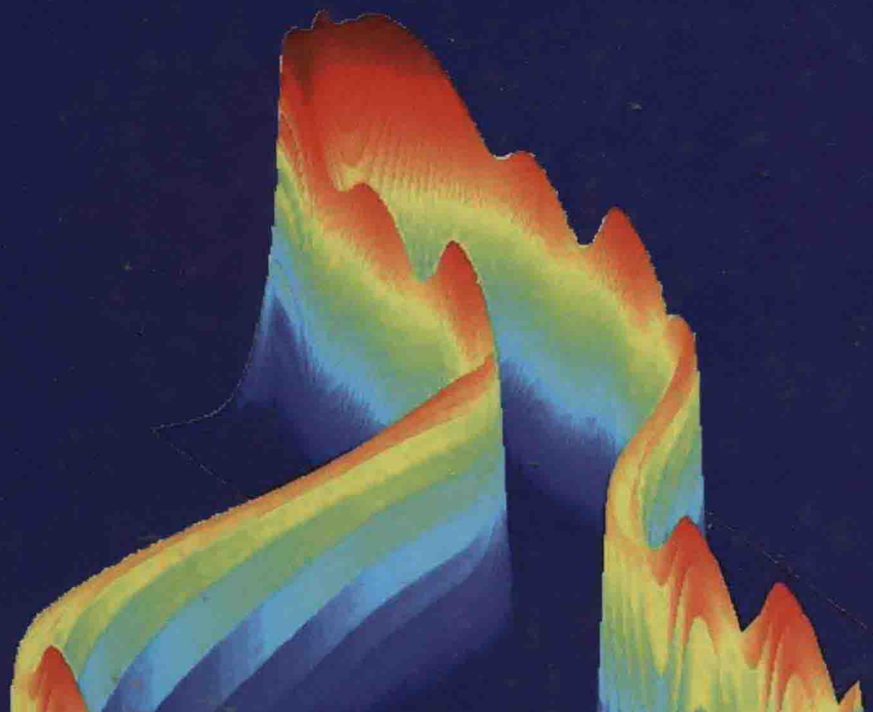
Computational Partial Differential Equations

Problems and Solutions



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Preface

The study of partial differential equations is a fundamental subject area of mathematics which links important strands of pure mathematics to applied and computational mathematics. Indeed partial differential equations are ubiquitous in almost all of the applications of mathematics where they provide a natural mathematical description of phenomena in the physical, natural and social sciences. Partial differential equations and their solutions exhibit rich and complex structures. Unfortunately, closed analytical expressions for their solutions can be found only in very special circumstances, and these are mostly of limited theoretical and practical interest. Thus, scientists and mathematicians have been naturally led to seeking techniques for the approximation of solutions. Indeed, the advent of digital computers has stimulated the incarnation of Computational Mathematics, much of which is concerned with the construction and the mathematical analysis of numerical algorithms for the approximate solution of partial differential equations. The efficient and reliable solution of partial differential equations (PDEs) plays an essential role in a very large number of applications in business, engineering and science, ranging from the modelling of financial markets through to the prediction of complex fluid flows. This paper presents a discussion of alternative approaches to the fast solution of elliptic and parabolic PDEs based upon the use of parallel, adaptive and multilevel algorithms. Mesh adaptivity is essential to ensure that the solution is approximated to different local resolutions across the domain according to its local properties, whilst the multilevel algorithms ensure that the computational time to solve the resulting finite element equations is proportional to the number of unknowns. Applying these techniques efficiently on parallel computer architectures leads to significant practical problems.

In this book we discuss the efficient numerical solution of elliptic and parabolic partial differential equations (PDEs) based upon the combination of three core ingredients: multilevel solvers, mesh adaptivity and parallel computing. Each of these topics have been actively and broadly studied in their own right in recent years and so it would be unrealistic to attempt to provide a comprehensive introduction to any of them in a short paper such as this. It is clear however that the use of any of these techniques within a computational algorithm has the potential to yield significant enhancements in computational efficiency. Combining any two of these approaches allows the possibility of further efficiency gains at the expense of increased programming complexity, whilst the use of all three has the potential for yet more improvement in performance provided that a number of challenging technical difficulties can be overcome successfully. In this book we present some of these technical issues and discuss the author's experiences in attempting to address them.

The present book focuses on some of the most exciting and promising mathematical ideas in these fields, and those branches of partial differential equations theory that provide a source of physically relevant and mathematically hard problems to stimulate future development. This book concentrates on the numerical solution of partial differential equations commonly encountered in Engineering Sciences. Finite difference and finite element methods are used to solve problems in heat flow, wave propagation, vibrations, fluid mechanics, hydrology, and solid mechanics. The chapters emphasize the systematic generation of numerical methods for elliptic, parabolic, and hyperbolic problems, and the analysis of their stability, accuracy, and convergence properties.

—*Editor*

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Chapter 1

Introduction

In mathematics, a partial differential equation (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations, which deal with functions of a single variable and its derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, or elasticity. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

Introduction

Partial differential equations (PDEs) are equations that involve rates of change with respect to continuous variables. The position of a rigid body is specified by six numbers, but the configuration of a fluid is given by the continuous distribution of several parameters, such as the temperature, pressure, and so forth.

The dynamics for the rigid body take place in a finite-dimensional configuration space; the dynamics for the fluid occur in an infinite-dimensional configuration space. This distinction usually makes PDEs much harder to solve than ordinary differential equations (ODEs), but here again there will be simple solutions for linear problems. Classic domains where PDEs are used include acoustics, fluid flow, electrodynamics, and heat transfer.

A partial differential equation (PDE) for the function $u(x_1,\cdots,x_n)$ is an equation of the form

$$F\left(x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1\partial x_1},\ldots,\frac{\partial^2 u}{\partial x_1\partial x_n},\ldots\right)=0.$$

If F is a linear function of u and its derivatives, then the PDE is called linear. Common examples of linear PDEs include the heat equation, the wave equation, Laplace's equation, Helmholtz equation, Klein–Gordon equation, and Poisson's equation.

A relatively simple PDE is

$$\frac{\partial u}{\partial x}(x,y)=0.$$

This relation implies that the function $u(x,y)$ is independent of x. However, the equation gives no information on the function's dependence on the variable y. Hence the general solution of this equation is

$$u(x,y)=f(y),$$

where f is an arbitrary function of y. The analogous ordinary differential equation is

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x)=0,$$

which has the solution

$$u(x)=c,$$

where c is any constant value. These two examples illustrate that general solutions of ordinary differential equations (ODEs) involve arbitrary constants, but solutions of PDEs involve arbitrary functions. A solution of a PDE is generally not unique; additional conditions must generally be specified on the boundary of the region where the solution is defined. For instance, in the simple example above, the function $f(y)$ can be determined if u is specified on the line $x = 0$.

Existence and Uniqueness

Although the issue of existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer with the Picard–Lindelöf theorem, that is far from the case for partial differential equations. The Cauchy–Kowalevski theorem states that the Cauchy problem for any partial differential equation whose coefficients are analytic in the unknown function and its derivatives, has a locally unique analytic solution. Although this result might appear to settle the existence and uniqueness of solutions, there are examples of linear partial differential equations whose coefficients have derivatives of all orders (which are nevertheless not analytic) but which have no solutions at all. Even if the solution of a partial differential equation exists and is unique, it may nevertheless have undesirable properties. The mathematical study of these questions is usually in the more powerful context of weak solutions.

An example of pathological behaviour is the sequence of Cauchy problems (depending upon n) for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

with boundary conditions

$$u(x,0)=0,$$

$$\frac{\partial u}{\partial y}(x,0)=\frac{\sin(nx)}{n},$$

where n is an integer. The derivative of u with respect to y approaches 0 uniformly in x as n increases, but the solution is

$$u(x,y)=\frac{\sinh(ny)\sin(nx)}{n^2}.$$

This solution approaches infinity if nx is not an integer multiple of π for any non-zero value of y. The Cauchy problem for the Laplace equation is called *ill-posed* or *not well posed*, since the solution does not depend continuously upon the data of the problem. Such ill-posed problems are not usually satisfactory for physical applications.

Notation

In PDEs, it is common to denote partial derivatives using subscripts. That is:

$$u_x=\frac{\partial u}{\partial x}$$

$$u_{xy}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Especially in physics, del (∇) is often used for spatial derivatives, and $\dot{u},\ddot{u}$ for time derivatives. For example, the wave equation (described below) can be written as

$$\ddot{u} = c^2\nabla^2 u$$

or

$$\ddot{u} = c^2\Delta u$$

where Δ is the Laplace operator.

Examples

Heat Equation in One Space Dimension: The equation for conduction of heat in one dimension for a homogeneous body has

$$u_t = \alpha u_{xx}$$

where $u(t,x)$ is temperature, and α is a positive constant that describes the rate of diffusion. The Cauchy problem for this equation consists in specifying $u(0, x)= f(x)$, where $f(x)$ is an arbitrary function.

General solutions of the heat equation can be found by the method of separation of variables. Some examples appear in the heat equation article. They are examples of Fourier series for periodic f and Fourier transforms for non-periodic f. Using the Fourier transform, a general solution of the heat equation has the form

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)e^{-\alpha\xi^2 t} e^{i\xi x} d\xi,$$

where F is an arbitrary function. To satisfy the initial condition, F is given by the Fourier transform of f, that is

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx.$$

If f represents a very small but intense source of heat, then the preceding integral can be approximated by the delta distribution, multiplied by the strength of the source. For a source whose strength is normalized to 1, the result is

$$F(\xi) = \frac{1}{\sqrt{2\pi}},$$

and the resulting solution of the heat equation is

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha\xi^2 t} e^{i\xi x} d\xi.$$

This is a Gaussian integral. It may be evaluated to obtain

$$u(t,x) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t} \right).$$

This result corresponds to the normal probability density for x with mean 0 and variance $2\alpha t$. The heat equation and similar diffusion equations are useful tools to study random phenomena.

Wave Equation in One Spatial Dimension

The wave equation is an equation for an unknown function $u(t, x)$ of the form

$$u_{tt} = c^2 u_{xx}.$$

Here u might describe the displacement of a stretched string from equilibrium, or the difference in air pressure in a tube, or the magnitude of an electromagnetic field in a tube, and c is a number that corresponds to the velocity of the wave. The Cauchy problem for this equation consists in prescribing the initial displacement and velocity of a string or other medium:

$$u(0,x) = f(x),$$
$$u_t(0,x) = g(x),$$

where f and g are arbitrary given functions. The solution of this problem is given by d'Alembert's formula:

$$u(t,x) = \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy.$$

This formula implies that the solution at (t,x) depends only upon the data on the segment of the initial line that is cut out by the characteristic curves

$$x - ct = \text{constant}, \quad x + ct = \text{constant},$$

that are drawn backwards from that point. These curves correspond to signals that propagate with velocity c forward and backward. Conversely, the influence of the data at any given point on the initial line propagates with the finite velocity c: there is no effect outside a triangle through that point whose sides are characteristic curves. This behaviour is very different from the solution for the heat equation, where the effect of a point source appears (with small amplitude) instantaneously at every point in space. The solution given above is also valid if $t < 0$, and the explicit formula shows that the solution depends smoothly upon the data: both the forward and backward Cauchy problems for the wave equation are well-posed.

Generalised Heat-Like Equation in One Space Dimension

Where heat-like equation means equations of the form:

$$\frac{\partial u}{\partial t} = \hat{H}u + f(x,t)u + g(x,t)$$

where $\hat{H}$ is a Sturm–Liouville operator (However it should be noted this operator may in fact be of the form

$$\frac{1}{w(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right)$$

where $w(x)$ is the weighting function with respect to which the eigenfunctions of $\hat{H}$ are orthogonal) in the x coordinate. Subject to the boundary conditions:

$$u(x,0)=h(x).$$

Then:

If:

$$\hat{H}X_n = \lambda_n X_n$$

$$X_n(a)=X_n(b)=0$$

$$\dot{a}_n(t)-\lambda_n a_n(t)-\sum_m (X_n f(x,t),X_m)a_m(t)=(g(x,t),X_n)$$

$$a_n(0)=\frac{(h(x),X_n)}{(X_n,X_n)}$$

$$u(x,t)=\sum_n a_n(t)X_n(x)$$

where

$$(f,g)=\int_a^b f(x)g(x)w(x)dx.$$

Spherical Waves

Spherical waves are waves whose amplitude depends only upon the radial distance r from a central point source. For such waves, the three-dimensional wave equation takes the form

$$u_{tt}=c^2\left[u_{rr}+\frac{2}{r}u_r\right].$$

This is equivalent to

$$(ru)_{tt}=c^2\left[(ru)_{rr}\right],$$

and hence the quantity ru satisfies the one-dimensional wave equation. Therefore a general solution for spherical waves has the form

$$u(t,r)=\frac{1}{r}\left[F(r-ct)+G(r+ct)\right],$$

where F and G are completely arbitrary functions. Radiation from an antenna corresponds to the case where G is identically zero. Thus the wave form transmitted from an antenna has no distortion in time: the only distorting factor is $1/r$. This feature of undistorted propagation of waves is not present if there are two spatial dimensions.

Laplace Equation in Two Dimensions

The Laplace equation for an unknown function of two variables φ has the form

$$\varphi_{xx} + \varphi_{yy} = 0.$$

Solutions of Laplace's equation are called harmonic functions.

Connection with Holomorphic Functions

Solutions of the Laplace equation in two dimensions are intimately connected with analytic functions of a complex variable (a.k.a. holomorphic functions): the real and imaginary parts of any analytic function are conjugate harmonic functions: they both satisfy the Laplace equation, and their gradients are orthogonal. If f=u+iv, *then the Cauchy–Riemann equations state that*

$$u_x = v_y, \quad v_x = -u_y,$$

and it follows that

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

Conversely, given any harmonic function in two dimensions, it is the real part of an analytic function, at least locally. Details are given in Laplace equation.

A Typical Boundary Value Problem

A typical problem for Laplace's equation is to find a solution that satisfies arbitrary values on the boundary of a domain. For example, we may seek a harmonic function that takes on the values u(θ) *on a circle of radius one. The solution was given by Poisson:*

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\theta')} u(\theta')d\theta'.$$

Petrovsky (1967, p. 248) shows how this formula can be obtained by summing a Fourier series for φ. If $r < 1$, the derivatives of φ may be computed by differentiating under the integral sign, and one can verify that φ is analytic, even if u is continuous but not necessarily differentiable. This behaviour is typical for solutions of elliptic partial differential equations: the solutions may be much more smooth than the boundary data. This is in contrast to solutions of the wave equation, and more general hyperbolic partial differential equations, which typically have no more derivatives than the data.

Euler–Tricomi Equation

The Euler–Tricomi equation is used in the investigation of transonic flow.

$$u_{xx} = xu_{yy}.$$

Advection Equation

The advection equation describes the transport of a conserved ..... ψ in a velocity field u = (u, v, w). It is:

$$\psi_t + (u\psi)_x + (v\psi)_y + (w\psi)_z = 0.$$

If the velocity field is solenoidal (that is, $\nabla \cdot$ u), then the equation may be simplified to

$$\psi_t + u\psi_x + v\psi_y + w\psi_z = 0.$$

In the one-dimensional case where u is not constant and is equal to ψ, the equation is referred to as Burgers' equation.

Ginzburg–Landau Equation

The Ginzburg–Landau equation is used in modelling superconductivity. It is

$$iu_t + pu_{xx} + q|u|^2 u = i\gamma u$$

where $p,q \in$ C and $\gamma \in$ R are constants and i is the imaginary unit.

The Dym equation

The Dym equation is named for Harry Dym and occurs in the study of solitons. It is

$$u_t = u^3 u_{xxx}.$$

Initial-Boundary Value Problems

Many problems of mathematical physics are formulated as initial-boundary value problems.

Vibrating String

If the string is stretched between two points where x=*0 and* x=L *and* u *denotes the amplitude of the displacement of the string, then* u *satisfies the one-dimensional wave equation in the region where* *0* < x < L *and* t *is unlimited. Since the string is tied down at the ends,* u *must also satisfy the boundary conditions*