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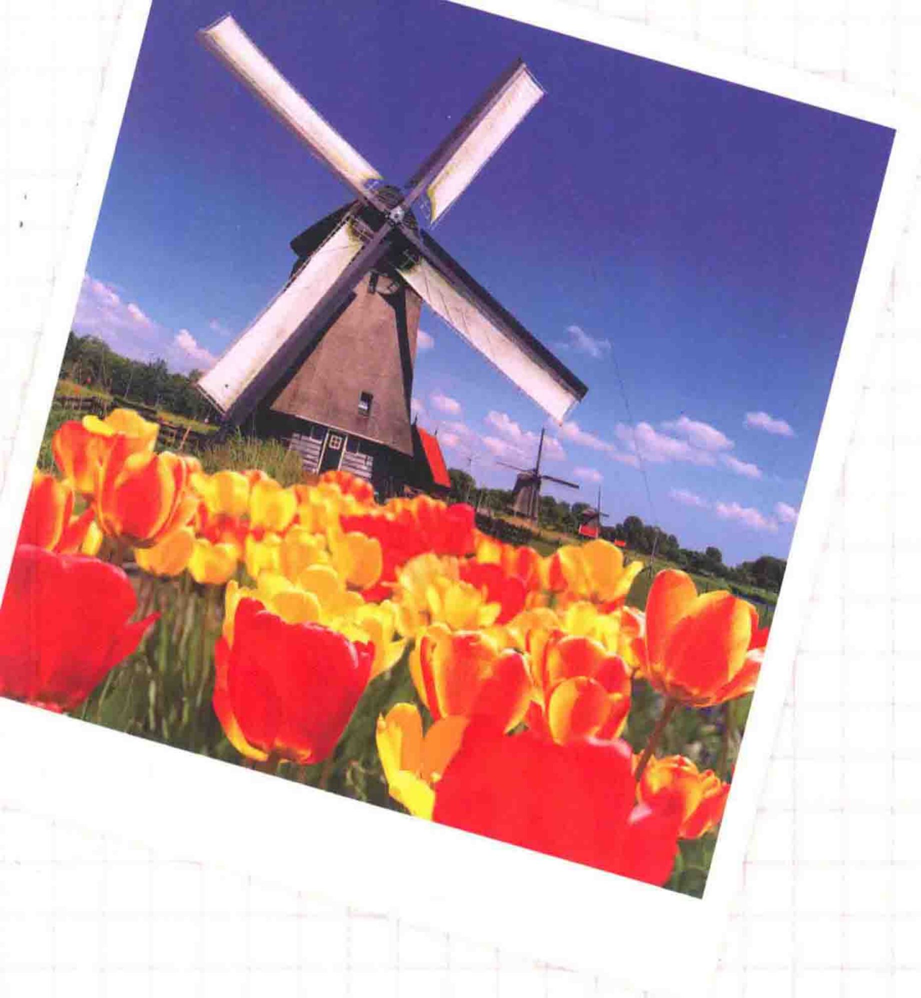
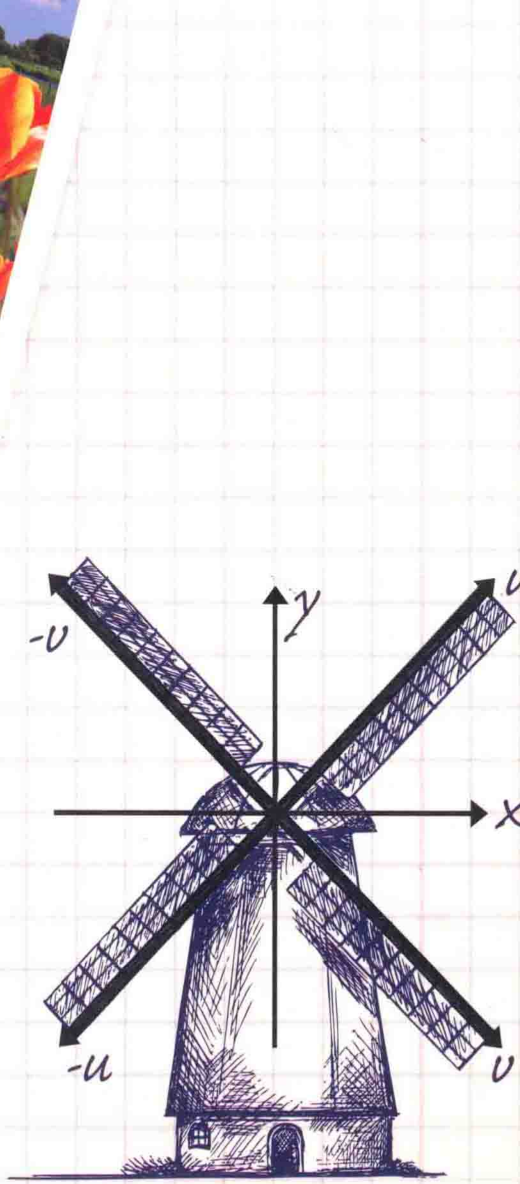


KULDEEP SINGH

linear algebra

step by step



Linear Algebra

Step by Step

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Preface

My interest in mathematics began at school. I am originally of Sikh descent, and as a young child often found English difficult to comprehend, but I discovered an affinity with mathematics, a universal language that I could begin to learn from the same start point as my peers.

Linear algebra is a fundamental area of mathematics, and is arguably the most powerful mathematical tool ever developed. It is a core topic of study within fields as diverse as business, economics, engineering, physics, computer science, ecology, sociology, demography and genetics. For an example of linear algebra at work, one need look no further than the Google search engine, which relies on linear algebra to rank the results of a search with respect to relevance.

My passion has always been to teach, and I have held the position of Senior Lecturer in Mathematics at the University of Hertfordshire for over twenty years, where I teach linear algebra to entry level undergraduates. I am also the author of *Engineering Mathematics Through Applications*, a book that I am proud to say is used widely as the basis for undergraduate studies in many different countries. I also host and regularly update a website dedicated to mathematics.

At the University of Hertfordshire we have over one hundred mathematics undergraduates. In the past we have based our linear algebra courses on various existing textbooks, but in general students have found them hard to digest; one of my primary concerns has been in finding rigorous, yet accessible textbooks to recommend to my students. Because of the popularity of my previously published book, I have felt compelled to construct a book on linear algebra that bridges the considerable divide between school and undergraduate mathematics.

I am somewhat fortunate in that I have had so many students to assist me in evaluating each chapter. In response to their reactions, I have modified, expanded and added sections to ensure that its content entirely encompasses the ability of students with a limited mathematical background, as well as the more advanced scholars under my tutelage. I believe that this has allowed me to create a book that is unparalleled in the simplicity of its explanation, yet comprehensive in its approach to even the most challenging aspects of this topic.

Level

This book is intended for first- and second-year undergraduates arriving with average mathematics grades. Many students find the transition between school and undergraduate mathematics difficult, and this book specifically addresses that gap and allows seamless progression. It assumes limited prior mathematical knowledge, yet also covers difficult material and answers tough questions through the use of clear explanation and a wealth of illustrations. The emphasis of the book is on students learning for themselves by gradually absorbing clearly presented text, supported by patterns, graphs and associated questions. The text allows the student to gradually develop an understanding of a topic, without the need for constant additional support from a tutor.

Pedagogical Issues

The strength of the text is in the large number of examples and the step-by-step explanation of each topic as it is introduced. It is compiled in a way that allows distance learning, with explicit solutions to all of the set problems freely available online <http://www.oup.co.uk/companion/singh>. The miscellaneous exercises at the end of each chapter comprise questions from past exam papers from various universities, helping to reinforce the reader's confidence. Also included are short historical biographies of the leading players in the field of linear algebra. These are generally placed at the beginning of a section to engage the interest of the student from the outset.

Published textbooks on this subject tend to be rather static in their presentation. By contrast, my book strives to be significantly more dynamic, and encourages the engagement of the reader with frequent question and answer sections. The question–answer element is sprinkled liberally throughout the text, consistently testing the student's understanding of the methods introduced, rather than requiring them to remember by rote.

The simple yet concise nature of its content is specifically designed to aid the weaker student, but its rigorous approach and comprehensive manner make it entirely appropriate reference material for mathematicians at every level. Included in the online resource will be a selection of MATLAB scripts, provided for those students who wish to process their work using a computer.

Finally, it must be acknowledged that linear algebra can appear abstract when first encountered by a student. To show off some of its possibilities and potential, interviews with leading academics and practitioners have been placed between chapters, giving readers a taste of what may be to come once they have mastered this powerful mathematical tool.

Acknowledgements

I would particularly like to thank Timothy Peacock for his signficant help in improving this text. In addition I want to thank Sandra Starke for her considerable contribution in making this text accessible.

Thanks too to the OUP team, in particular Keith Mansfield, Viki Mortimer, Smita Gupta and Clare Charles.

Dedication

To Shaheed Bibi Paramjit Kaur

Contents

1 Linear Equations and Matrices

SECTION 1.1 Systems of Linear Equations

By the end of this section you will be able to

- solve a linear system of equations
- plot linear graphs and determine the type of solutions

1.1.1 Introduction to linear algebra

We are all familiar with simple one-line equations. An equation is where two mathematical expressions are defined as being equal. Given $3x = 6$, we can almost intuitively see that x must equal 2.

However, the solution isn't always this easy to find, and the following example demonstrates how we can extract information embedded in more than one line of information.

Imagine for a moment that John has bought two ice creams and two drinks for £3.00.

How much did John pay for each item?

Let $x =$ cost of ice cream and $y =$ cost of drink, then the problem can be written as

$$2x + 2y = 3$$

At this point, it is impossible to find a unique value for the cost of each item. However, you are then told that Jane bought two ice creams and one drink for £2.50. With this additional information, we can model the problem as a **system of equations** and look for unique values for the cost of ice creams and drinks. The problem can now be written as

$$\begin{aligned} 2x + 2y &= 3 \\ 2x + y &= 2.5 \end{aligned}$$

Using a bit of guesswork, we can see that the only sensible values for x and y that satisfy both equations are $x = 1$ and $y = 0.5$. Therefore an ice cream must have cost £1.00 and a drink £0.50.

Of course, this is an extremely simple example, the solution to which can be found with a minimum of calculation, but larger systems of equations occur in areas like engineering, science and finance. In order to reliably extract information from multiple linear equations, we need linear algebra. Generally, the complex scientific, or engineering problem can be solved by using linear algebra on linear equations.

*What does the term **linear equation** mean?*
An equation is where two mathematical expressions are defined as being equal.

A linear equation is one where all the variables such as x, y, z have index (power) of 1 or 0 only, for example

$$x + 2y + z = 5$$

is a linear equation. The following are also linear equations:

$$x = 3;\ x + 2y = 5;\ 3x + y + z + w = -8$$

The following are *not* linear equations:

1. $x^2 - 1 = 0$
2. $x + y^4 + \sqrt{z} = 9$
3. $\sin(x) - y + z = 3$

Why not?
In equation (1) the index (power) of the variable x is 2, so this is actually a quadratic equation.
In equation (2) the index of y is 4 and z is 1/2. Remember, $\sqrt{z} = z^{1/2}$.
In equation (3) the variable x is an argument of the trigonometric function sine.

Note that if an equation contains an **argument** of trigonometric, exponential, logarithmic or hyperbolic functions then the equation is not linear.

A set of linear equations is called a **linear system**.

In this first course on linear algebra we examine the following questions regarding linear systems:

- Are there any solutions?
- Does the system have no solution, a unique solution or an infinite number of solutions?
- How can we find all the solutions, if they exist?
- Is there some sort of structure to the solutions?

Linear algebra is a systematic exploration of linear equations and is related to 'a new kind of arithmetic' called the **arithmetic of matrices** which we will discuss later in the chapter.

However, linear algebra isn't exclusively about solving linear systems. The tools of matrices and vectors have a whole wealth of applications in the fields of functional analysis and quantum mechanics, where inner product spaces are important. Other applications include optimization and approximation where the critical questions are:

1. Given a set of points, what's the best linear model for them?
2. Given a function, what's the best polynomial approximation to it?

To solve these problems we need to use the concepts of eigenvalues and eigenvectors and orthonormal bases which are discussed in later chapters.

In all of mathematics, the concept of linearization is critical because linear problems are very well understood and we can say a lot about them. For this reason we try to convert many areas of mathematics to linear problems so that we can solve them.

1.1.2 System of linear equations

We can plot linear equations on a graph. Figure 1.1 shows the example of the two linear equations we discussed earlier.

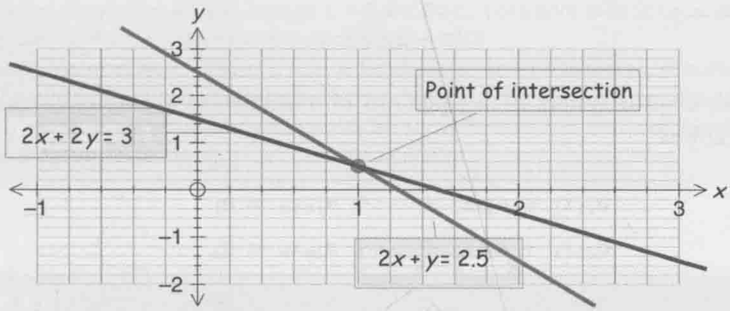


Figure 1.1

Figure 1.2 is an example of the linear equation, $x + y + 2z = 0$, in a 3d coordinate system.

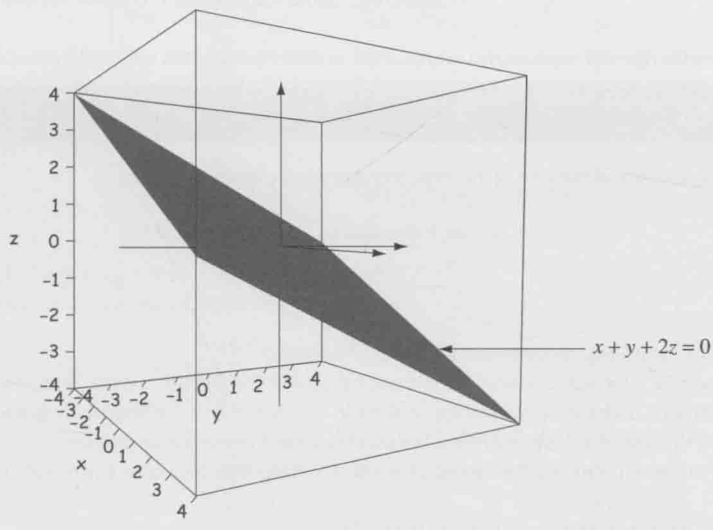


Figure 1.2

What do you notice about the graphs of linear equations?
They are straight lines in 2d and a plane in 3d. This is why they are called linear equations and the study of such equations is called **linear algebra**.

What does the term system of linear equations mean?
Generally a finite number of linear equations with a finite number of unknowns $x, y, z, w, \ldots$ is called a **system of linear equations** or just a **linear system**.

For example, the following is a linear system of three simultaneous equations with three unknowns x, y and z:

$$\begin{aligned} x + 2y - 3z &= 3 \\ 2x - y - z &= 11 \\ 3x + 2y + z &= -5 \end{aligned}$$

In general, a linear system of m equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is written mathematically as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \qquad (*)$$

where the coefficients a_{ij} and b_j represent real numbers. The unknowns $x_1, x_2, \ldots, x_n$ are **placeholders** for real numbers.

Linear algebra involves using a variety of methods for finding solutions to linear systems such as (*).

Example 1.1

Solve the equations about the cost of ice creams and drinks by algebraic means

$$\begin{aligned} 2x + 2y &= 3 \qquad (1) \\ 2x + y &= 2.5 \qquad (2) \end{aligned}$$

Solution

How do we solve these linear simultaneous equations, (1) and (2)?

Let's think about the information contained in these equations. The x in the first line represents the cost of an ice cream, so must have the same value as the x in the second line. Similarly, the y in the first line that represents the cost of a drink must have the same value as the y in the second line.

It follows that we can combine the two equations to see if together they offer any useful information.

How?

In this case, we subtract equation (2) from equation (1):

$$\begin{aligned} 2x + 2y &= 3 \qquad (1) \\ -(2x + y &= 2.5) \qquad (2) \\ \hline 0 + y &= 0.5 \end{aligned}$$

Note that the unknown x is eliminated in the last line which leaves $y = 0.5$.

What else do we need to find?

The other unknown x.

How?
By substituting $y = 0.5$ into equation (1):

$$2x + 2\,(0.5) = 3 \quad \text{implies that} \quad 2x + 1 = 3 \quad \text{gives} \quad x = 1$$

Hence the cost of an ice cream is £1 because $x = 1$ and the cost of a drink is £0.50 because $y = 0.5$; this is the solution to the given simultaneous equations (1) and (2).

This is also the **point of intersection**, (1, 0.5), of the graphs in Fig. 1.1. The procedure outlined in Example 1.1 is called the method of **elimination**. The values $x = 1$ and $y = 0.5$ is the solution of equations (1) and (2). In general, values which satisfy the above linear system are called the **solution** or the **solution set** of the linear system. Here is another example.

Example 1.2

Solve

$$\begin{aligned} 9x + 3y &= 6 \qquad (1) \\ 2x - 7y &= 9 \qquad (2) \end{aligned}$$

Solution
We need to find the values of x and y which satisfy both equations.
How?
Taking one equation from the other doesn't help us here, but we can multiply through either or both equations by a non-zero constant.

If we multiply equation (1) by 2 and (2) by 9 then in both cases the x coefficient becomes 18. Carrying out this operation we have

$$\begin{aligned} 18x + 6y &= 12 \quad \left[\text{multiplying equation (1) by 2}\right] \\ 18x - 63y &= 81 \quad \left[\text{multiplying equation (2) by 9}\right] \end{aligned}$$

How do we eliminate x from these equations?
To eliminate the unknown x we subtract these equations:

$$\begin{array}{r} 18x + 6y = 12 \\ -(18x - 63y = 81) \\ \hline \end{array}$$

$$\begin{aligned} 0 + [6 - (-63)]\,y &= 12 - 81 \quad \left[\text{subtracting}\right] \\ 69y &= -69 \quad \text{which gives } y = -1 \end{aligned}$$

We have $y = -1$.
What else do we need to find?
The value of the placeholder x.
How?
By substituting $y = -1$ into the given equation $9x + 3y = 6$:

$$\begin{aligned} 9x + 3\,(-1) &= 6 \\ 9x - 3 &= 6 \\ 9x &= 9 \text{ which gives } x = 1 \end{aligned}$$

(continued...)

Hence our solution to the linear system of (1) and (2) is

$$x = 1 \text{ and } y = -1$$

We can check that this is the solution to the given system, (1) and (2), by substituting these values, $x = 1$ and $y = -1$, into the equations (1) and (2).

Note that we can carry out the following operations on a linear system of equations:

1. Interchange any pair of equations.
2. Multiply an equation by a non-zero constant.
3. Add or subtract one equation from another.

By carrying out these steps 1, 2 and 3 we end up with a simpler linear system to solve, but with the same solution set as the original linear system. In the above case we had

$$\begin{aligned} 9x + 3y &= 6 \\ 2x - 7y &= 9 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 9x + 3y &= 6 \\ 69y &= -69 \end{aligned}$$

Of course, the system on the right hand side was much easier to solve. We can also use this method of elimination to solve three simultaneous linear equations with three unknowns, such as the one in the next example.

Example 1.3

Solve the linear system

$$\begin{aligned} x + 2y + 4z &= 7 && (1) \\ 3x + 7y + 2z &= -11 && (2) \\ 2x + 3y + 3z &= 1 && (3) \end{aligned}$$

Solution

What are we trying to find?

The values of x, y and z that satisfy all three equations (1), (2) and (3).

How do we find the values of x, y and z?

By elimination. To eliminate one of these unknowns, we first need to make the coefficients of x (or y or z) equal.

Which one?

There are three choices but we select so that the arithmetic is made easier, in this case it is x. Multiply equation (1) by 2 and then subtract the bottom equation (3):

$$\begin{aligned} 2x + 4y + 8z &= 14 && \text{[multiplying (1) by 2]} \\ -(2x + 3y + 3z &= 1) && (3) \\ \hline 0 + y + 5z &= 13 && \text{[subtracting]} \end{aligned}$$

Note that we have eliminated x and have the equation $y + 5z = 13$.
How can we determine the values of y and z from this equation?
We need another equation with only y and z.
How can we get this?
Multiply equation (1) by 3 and then subtract the second equation (2):

$$\begin{array}{rl} 3x + 6y + 12z = 21 & \text{[multiplying (1) by 3]} \\ -(3x + 7y + 2z = -11) & (2) \\ \hline 0 - y + 10z = 32 & \text{[subtracting]} \end{array}$$

Again there is no x and we have the equation $-y + 10z = 32$.
How can we find y and z?
We now solve the two simultaneous equations that we have obtained

$$\begin{aligned} y + 5z &= 13 \qquad (4) \\ -y + 10z &= 32 \qquad (5) \end{aligned}$$

We add equations (4) and (5), because $y + (-y) = 0$, which eliminates y.

$$0 + 15z = 45 \text{ gives } z = \frac{45}{15} = 3$$

Hence $z = 3$, but how do we find the other two unknowns x and y?
We first determine y by substituting $z = 3$ into equation (4) $y + 5z = 13$:

$$\begin{aligned} y + (5 \times 3) &= 13 \\ y + 15 &= 13 \qquad \text{which gives } y = -2 \end{aligned}$$

We have $y = -2$ and $z = 3$. We still need to find the value of last unknown x.
How do we find the value of x?
By substituting the values we have already found, $y = -2$ and $z = 3$, into the given equation $x + 2y + 4z = 7$ (1) :

$$x + (2 \times -2) + (4 \times 3) = 7 \quad \text{gives} \quad x = -1$$

Hence the solution of the given three linear equations is $x = -1$, $y = -2$ and $z = 3$.
We can illustrate the given equations in a three-dimensional coordinate system as shown in Fig. 1.3.

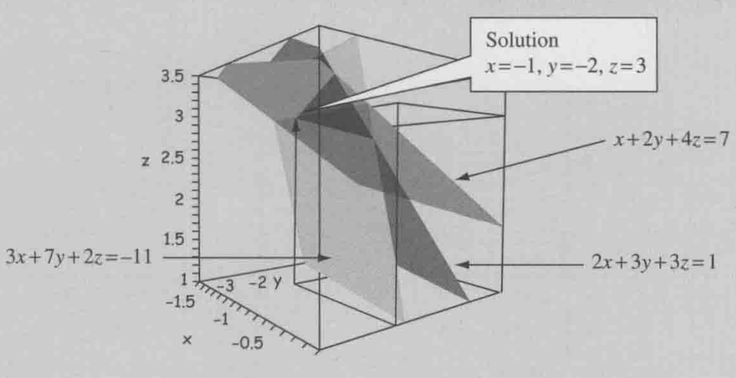


Figure 1.3

Each of the equations (1), (2) and (3) are represented by a plane in a three-dimensional system. The computer generated image above allows us to see where these planes lie with respect to each other. The coordinates of these planes is the solution of the system.

The aim of the above problem was to convert the given system into something simpler that could be solved. We had

$$\begin{aligned} x + 2y + 4z &= 7 \\ 3x + 7y + 2z &= -11 \\ 2x + 3y + 3z &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + 2y + 4z &= 7 \\ -y + 2z &= 32 \\ 15z &= 45 \end{aligned}$$

We will examine in detail m equations with n unknowns and develop a more efficient way of solving these later in this chapter.

1.1.3 Types of solutions

We now go back to evaluating a simple system of two linear simultaneous equations and discuss the case where we have no, or an infinite number of solutions. As stated earlier, one of the fundamental questions of linear algebra is how many solutions do we have of a given linear system.

Example 1.4

Solve the linear system

$$\begin{aligned} 2x + 3y &= 6 \quad &(1) \\ 4x + 6y &= 9 \quad &(2) \end{aligned}$$

Solution

How do we solve these equations?

Multiply (1) by 2 and then subtract equation (2):

$$\begin{aligned} 4x + 6y &= 12 \quad &[\text{Multiplying (1) by 2}] \\ -(4x + 6y &= 9) \quad &(2) \\ \hline 0 + 0 &= 3 \end{aligned}$$

But how can we have $0 = 3$*?*

A plot of the graphs of the given equations is shown in Fig. 1.4.

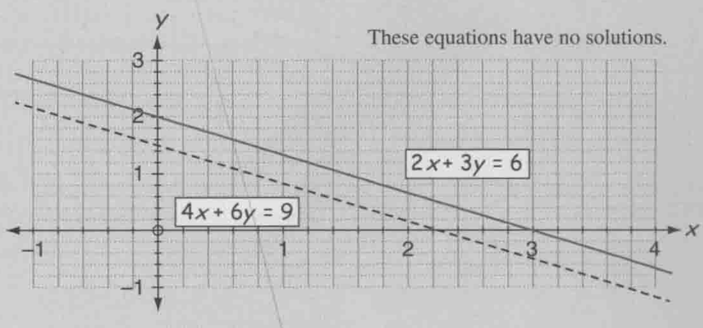


Figure 1.4

Can you see why there is no common solution to these equations?
The solution of the given equations would be the intersection of the lines shown in Fig. 1.4, but these lines are parallel so there is no intersection, therefore no solution.

By examining the given equations,

$$2x + 3y = 6 \qquad (1)$$
$$4x + 6y = 9 \qquad (2)$$

can you see why there is no solution?
If you multiply the first equation (1) by 2 we have

$$4x + 6y = 12$$

This is a contradiction.
Why?
Because we have

$$4x + 6y = 12$$
$$4x + 6y = 9 \qquad (2)$$

that is, $4x + 6y$ equals both 9 and 12. This is clearly impossible. Hence the given linear system has no solution.

A system that has no solution is called **inconsistent**. If the linear system has at least one solution then we say the system is **consistent**.

Can we have more than one solution?
Consider the following example.

Example 1.5

Graph the equations and determine the solution of this system:

$$2x + 3y = 6 \qquad (1)$$
$$4x + 6y = 12 \qquad (2)$$

Solution
The graph of the given equations is shown in Fig. 1.5.

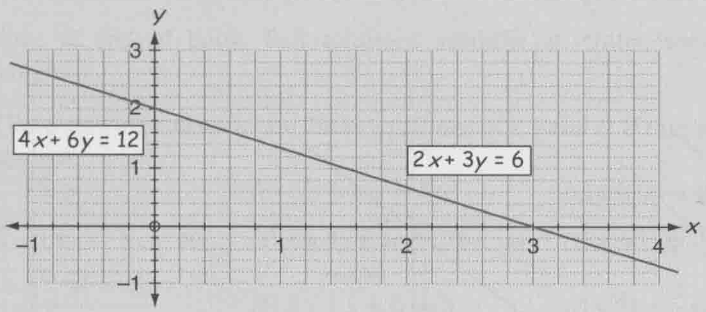


Figure 1.5

(continued...)

What do you notice?
Both the given equations produce exactly the same line; that is they coincide.
How many solutions do these equations have?
An infinite number of solutions, as you can see on the graph of Fig. 1.5. Any point on the line is a solution, and since there are an infinite number of points on the line we have an infinite number of solutions.
How can we write these solutions?
Let $x = a$ - where a is any real number - be a solution.
What then is y equal to?
Substituting $x = a$ into the given equation (1) yields

$$\begin{aligned} 2a + 3y &= 6 \qquad [2x + 3y = 6] \\ 3y &= 6 - 2a \\ y &= \frac{6-2a}{3} = \frac{6}{3} - \frac{2}{3}a = 2 - \frac{2}{3}a \end{aligned}$$

Hence if $x = a$ then $y = 2 - \frac{2}{3}a$.

The solution of the given linear system, (1) and (2), is $x = a$ and $y = 2 - 2a/3$ where a is any real number. You can check this by substituting various values of a. For example, if $a = 1$ then

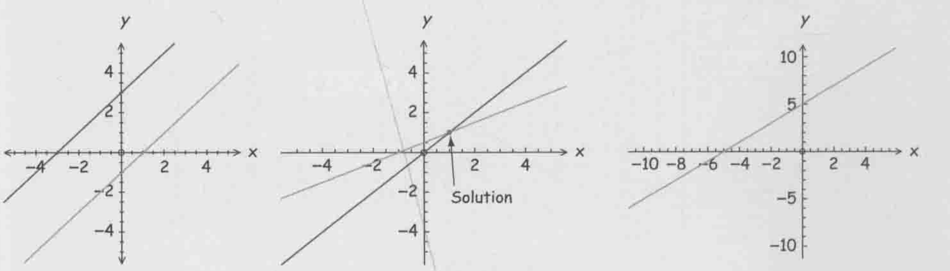
$$x = 1, \quad y = 2 - 2(1)/3 = 4/3$$

We can check that this answer is correct by substituting these values, $x = 1$ and $y = 4/3$, into equations (1) and (2):

$$\begin{aligned} 2(1) + 3(4/3) &= 2 + 4 = 6 \\ 4(1) + 6(4/3) &= 4 + 8 = 12 \end{aligned} \qquad \begin{bmatrix} 2x + 3y = 6 & (1) \\ 4x + 6y = 12 & (2) \end{bmatrix}$$

Hence our solution works. This solution $x = a$ and $y = 2 - 2a/3$ will satisfy the given equations for any real value of a.

The graphs in Fig. 1.6 represent the three possible solutions to a linear system with two unknowns.



(a) No solution (b) Unique solution (c) Infinite number of solutions

Figure 1.6