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An introduction to Harmonic Analysis on Semisimple Lie Groups

V. S. VARADARAJAN

AN INTRODUCTION TO

Harmonic analysis on semisimple Lie groups

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An introduction to harmonic analysis on semisimple Lie groups

See what a lovely shell,
Small and pure as a pearl,
Lying close to my foot,
Frail, but a work divine,
Made so fairily well
With delicate spire and whorl,
How exquisitely minute,
A miracle of design!
What is it? a learned man
Could give it a clumsy name.
Let him name it who can;
The beauty will be the same.

– Alfred, Lord Tennyson, *Maud* II, II.

Preface

A substantial part of the material covered in these notes formed the content of a course of lectures given during the Spring of 1985 in the Mathematics Institute of the University of Warwick, England. My aim was to introduce the aspiring graduate student to a beautiful and central part of mathematics, the representation theory of semisimple groups. This is of course a vast and active subject, bringing together at a fairly deep level algebra, geometry, analysis, and arithmetic. This is one of the reasons why it is difficult to get into, at least for the young student. I therefore made an attempt to keep the requirements minimal, and introduced the major themes of the subject by first working them out in the case of $SL(2, \mathbb{R})$. This approach has no claim to novelty: it has been done before, and there is the well-known book of S. Lang dealing only with $SL(2, \mathbb{R})$. I have, however, discussed a number of topics not treated by Lang, such as the Schwartz space, invariant eigendistributions, wave packets, and so on; in addition, I have included, wherever possible, indications of how these ideas may generalized to the context of a general semisimple Lie group.

The organization of the book is not always linear because I wanted to adhere closely to the lectures and preserve their freeflowing nature. As a result, the reader will often find references to matters that are not defined or are quite advanced, especially in Chapters 1–2. This should not discourage him (or her); my advice to the reader is to ignore it and proceed ahead, and come back to the difficult points later. Chapters 4–8 are essentially linear and can be worked through by a graduate student (late first year or early second year in an American university). I have included appendices on Functional Analysis and Lie theory that offer the reader some basic definitions, explanations of some concepts, and some historical perspective.

It is a great pleasure for me to express my gratitude to the Mathematics Institute of the University of Warwick and the Science and Engineering Research Council of the United Kingdom for inviting me to Warwick, and to the staff and faculty of the Institute for their wonderful hospitality. To thank Klaus and Annelise Schmidt adequately for what they did for my wife and me is impossible; it was only through their kindness and generosity that this visit become so memorable for us. Finally I want to thank the

Cambridge University Press and their editor Dr David Tranah for being really patient while I took my time preparing the final version of this manuscript.

Pacific Palisades, October 1986 V.S. Varadarajan

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Introduction

1.1 Aim

The aim of this book is to introduce advanced undergraduates or beginning graduate students to the subject of harmonic analysis on semisimple Lie groups. This involves doing a certain amount of representation theory for these groups, either implicitly or explicitly, because harmonic analysis is concerned mainly with expanding arbitrary functions (and generalized functions) on a group as a series or integral of functions which occur as matrix elements of irreducible representations of the group. Nevertheless this is not a book on representation theory. As long as the group is compact, the harmonic analysis point of view is not very prominent; but for noncompact groups the behaviour of the matrix elements at infinity becomes critical, and the analysis becomes decisive. Thus, although representation theory and harmonic analysis have a lot in common, the two subjects are not quite the same; and the differences will become clear to the reader when both themes have been developed to a certain extent. In this introductory chapter I shall discuss briefly a number of sources of motivation for studying representations and harmonic analysis, whose diversity and wide-ranging nature show that our subject is much more central than it seems at first sight.

1.2 Some definitions

A *representation* of a group G is a homomorphism of G into the general linear group $GL(V)$ of a complex finite-dimensional vector space V; the representation is said to be *in* V. If 0 and V are the only linear subspaces of V stable under the representation π (i.e., left invariant by $\pi(g)$ for all $g \in G$), then π is said to be *irreducible*. Representations π_j in V_j $(j = 1, 2)$ are *equivalent* if there is a linear isomorphism $T(V_1 \to V_2)$ such that $\pi_2(g) = T\pi_1(g)T^{-1}$ for all $g \in G$. If π_j are representations of G on V_j $(j = 1, \dots, m)$ their direct sum $\pi = \pi_1 \oplus \cdots \oplus \pi_m$ and tensor product $\pi' = \pi_1 \otimes \cdots \otimes \pi_m$ are the representations defined respectively in $V = V_1 \oplus \cdots \oplus V_m$ and $V' = V_1 \otimes \cdots \otimes V_m$ by

$$\left.\begin{aligned} \pi(g) &= \pi_1(g) \oplus \cdots \oplus \pi_m(g) \\ \pi'(g) &= \pi_1(g) \otimes \cdots \otimes \pi_m(g) \end{aligned}\right\} \quad (g \in G)$$

The dual (or contragradient) of a representation π in V is the representation

π^* in the dual V^* of V defined by

$$(v, \pi^*(g)v^*) = (\pi(g^{-1})v, v^*) \quad (v \in V, v^* \in V^*, g \in G)$$

Actually, the category of finite-dimensional representations, with $\oplus$, $\otimes$, * defined as above, is adequate only for problems involving finite groups. For infinite groups, it is necessary to impose additional restrictions such as continuity, rationality, and so on, as well as to consider infinite-dimensional representations. We begin by looking at some of the most common examples.

Finite groups. As we remarked above, the category of representations in finite-dimensional vector spaces is the natural one to work with, but the restriction to complex vector spaces is not reasonable. Most applications, in physics and chemistry, for example, deal with complex representations; but for a general theory the underlying fields should be arbitrary [S1].

Algebraic groups. In concrete terms these are groups of matrices defined by polynomial conditions on their entries. Typical examples are the unimodular group, i.e., the group of matrices of determinant 1, the orthogonal group, the symplectic group, and so on. For such a group defined over (i.e., with entries from) an algebraically closed field k, it is natural to work only with finite-dimensional representations π which are *rational*; here rational means the entries of $\pi(g)$ relative to a basis of V are polynomial functions of the entries of g and $\det(g)^{-1}$.

Topological groups. Impulses from functional analysis and quantum physics were very much responsible for a systematic study of representations in infinite-dimensional spaces. The groups considered are topological and the vector spaces usually complete and locally convex. If G is a topological group and V is a complete locally convex space, the homomorphism π of G into the group of invertible automorphisms of V is a *representation* if the map $(g, v) \mapsto \pi(g)(v)$ of $G \times V$ into V is continuous. Important special cases are when V is a Banach space and G is locally compact. If V is a Hilbert space and each $\pi(g)$ is unitary, π is called a *unitary representation.* For a general π, irreducibility now means that 0 and V are the only *closed* linear subspaces stable under π; equivalence of π_1 and π_2 is defined as before, but with T required now to be a topological linear isomorphism; if π_1 and π_2 are unitary, and T is unitary, π_1 and π_2 are said to be *unitary equivalent*. The set of equivalence classes of irreducible unitary representations of G is written $\hat{G}$ and is called the *unitary dual* of G. One can define infinite (orthogonal) direct sums in the category of unitary representations. The definition of tensor products for infinite-dimensional representations is somewhat technically involved since tensor product is a technically complicated notion for topological vector spaces; we shall not make any serious use of this.

1.3 Classical invariant theory

The geometric invariant theory of classical geometers was one of the first examples of an important context where representation theory entered in a nontrivial way. Here $G = SL(n+1, \mathbb{C})$ and one starts with a rational representation of the algebraic group G in a complex vector space V. The action of G on V gives rise to an action on the projective space $\mathbb{P}(V)$ of V. The problem of invariant theory is that of describing the orbit space $G\backslash\mathbb{P}(V)$[MF]. This leads almost immediately to the study of the action of G on the rings of functions on $\mathbb{P}(V)$. Let R be the graded ring of polynomials on V, and $\bar{R} = R^G$ be the graded subring of G-invariant polynomials. The first step in the description of $G\backslash\mathbb{P}(V)$ is the study of the following question:

$$\text{Is } \bar{R} \text{ finitely generated?} \qquad (*)$$

Hilbert proved, at the beginning of his epoch-making work on invariant theory, that for $G = SL(n+1, \mathbb{C})$ the answer to $(*)$ is affirmative. This was eventually extended to all complex semisimple groups G by Hermann Weyl who obtained it as a consequence of his famous theorem that all rational representations of any complex semisimple group are completely reducible, i.e., direct sums of irreducible representations. Weyl's theorem is one of the deepest and most important in the finite-dimensional representation theory of semisimple groups, and we shall discuss it briefly in the next chapter.

When the questions of geometric invariant theory were examined by Mumford in the 1960s Chevalley had already developed the theory of semisimple algebraic groups over any algebraically closed field k; and Mumford's investigations led naturally to the question of finite generation of $\bar{R} = R^G$ where R is the graded ring of polynomials on a vector space V on which we have a rational representation of the algebraic semisimple group G. Unfortunately Weyl's method fails when $\operatorname{char}(k) > 0$; representations of G are not in general completely reducible when k has positive characteristic. Nevertheless Mumford conjectured that all rational representations of the semisimple group G over an arbitrary algebraically closed field k possess the following property (M):

> if $v \neq 0$ is a vector in the space V of the given representation and v is invariant under G, there is a nonzero homogeneous polynomial f on V invariant under G such that $f(v) \neq 0$.

The property (M) is equivalent to complete irreducibility when $\operatorname{char}(k) = 0$;

its validity for general k implies $\bar{R}$ is finitely generated. Mumford's conjecture was proved by Haboush in 1975 [Hb]. These results have been the beginning of new progress in representation theory and geometric invariant theory [MF]. For the classical theory there are of course many references; in addition to Weyl's great classic [W1] the reader may consult Schur's lectures [Sc].

1.4 Quantum mechanics and unitary representations

We now turn to a completely different source of problems in which unitary representations appear prominently. The group G is now the symmetry group of a quantum-mechanical system and one is interested in a description of the system that is covariant under G. Now, any quantum-mechanical description requires the introduction of a complex Hilbert space $\mathscr{H}$; the physical interpretation consists in identifying the orthocomplemented lattice $\mathscr{L}(\mathscr{H})$ of the closed linear subspaces of $\mathscr{H}$ with the logic of experimentally verifiable propositions of the system [V1]. The requirement of covariance means there is a homomorphism σ of G into the group of automorphisms of $\mathscr{L}(\mathscr{H})$. Now it can be proved that any automorphism σ of $\mathscr{L}(\mathscr{H})$ is induced in the obvious way by a unitary or antiunitary operator, determined uniquely up to a multiplicative constant of absolute value 1. Under mild assumptions on G and σ it can be shown that σ is induced by a *projective unitary representation* of G, i.e. a unitary representation of *an extension of* G *by the group* T *complex numbers of absolute value* 1. This representation is obviously an important invariant of the system. For suitable G one can show that σ is induced by a unitary representation of its simply connected covering group $\tilde{G}$. This is the case when G is the group of automorphisms of Euclidean or Minkowskian affine space-time (however, this is *not* the case for the group of automorphisms of Galilean space-time).

If G is the group of automorphisms of an affine space with the structure of Minkowskian space-time, G can be written as a semidirect product $A \ltimes H$ where A is the four-dimensional group of space-time translations, $H = SL(2, \mathbb{C})$, and H acts linearly on A via the Lorentz transformations. In any description of a quantum-mechanical system consistent with special relativity there will thus appear a unitary representation of G. For instance, if the system is that of a free elementary particle, it is natural to expect this representation to be irreducible, and to expect further that it will tell us everything about this free particle. Thus the free relativistic elementary particles are in one-one correspondence with a certain subset of $\hat{G}$. Now there is a general method, due to Mackey, for determining the irreducible unitary representations of such (and even more general) locally compact semidirect products. This method, applied in the present situation, leads in a simple and natural manner to the classification of the particles in terms of their mass and spin [V1] [Ma1].

It is not always the case that the symmetry groups are locally compact. The *gauge groups* occurring in the theory of gauge fields are infinite-dimensional, and the representation theory of these and more general groups is quite active now, although not yet in any definitive state [K] [PS].

1.5 Classical Fourier analysis. Plancherel and Poisson formulae

The starting point of Fourier analysis is the idea that a more or less arbitrary function can be expanded as a 'linear combination' of the exponentials. The basic objective of the theory is to define the *Fourier transform*; the transform of a function (or a generalized function) shows how it is made up of its harmonic constituents. We shall now explain briefly the point of view of the theory of unitary representations that allows us to understand and generalize these classical themes.

Fourier series deal with functions on the torus $\mathbb{T}^n$ with coordinates $\theta = (\theta_1, \ldots, \theta_n)$. We introduce the Hilbert space $L^2(\mathbb{T}^n) = L^2(\mathbb{T}^n, d\theta)$, $d\theta = d\theta_1 \cdots d\theta_n$, $\int d\theta = 1$. For $\phi \in \mathbb{T}^n$ we define the linear operator $\lambda(\phi)$ on $L^2(\mathbb{T}^n)$ by

$$(\lambda(\phi)f)(\theta) = f(-\phi + \theta)$$

The $\lambda(\phi)$ are unitary, and it is easy to show that $\lambda(\phi \to \lambda(\phi))$ is a unitary representation of $\mathbb{T}^n$, the so-called *regular representation* of $\mathbb{T}^n$. The *irreducible* unitary representations of $\mathbb{T}^n$ are precisely all the *characters*

$$\chi_m: \theta \to \exp 2\pi i(m_1\theta_1 + \cdots + m_n\theta_n) \quad (m = (m_1, \ldots, m_n) \in \mathbb{Z}^n)$$

The functions χ_m are in $L^2(\mathbb{T}^n)$; the one-dimensional subspaces $\mathbb{C}\cdot\chi_m$ are stable under λ, and the restriction of λ to $\mathbb{C}\cdot\chi_{-m}$ is equivalent to the one-dimensional representation χ_m. The orthogonal direct sum decomposition

$$L^2(\mathbb{T}^n) = \bigoplus_m \mathbb{C}\cdot\chi_{-m}$$

shows that λ is equivalent to the infinite (orthogonal) direct sum of the χ_m ($m \in \mathbb{Z}^n$), each taken only once. We shall now see that the Fourier transform operator leads to the explicit 'diagonalization' of λ. For any $f \in L^2(\mathbb{T}^n)$ define its Fourier transform $\hat{f} = \mathscr{F} f$ by

$$\hat{f}(m) = (f, \chi_{-m}) \quad (m \in \mathbb{Z}^n)$$

where $(\cdot, \cdot)$ is the scalar product. Then $\hat{f}$ is a function on $\mathbb{Z}^n$. If we equip $\mathbb{Z}^n$ with the counting measure and introduce the Hilbert space $L^2(\mathbb{Z}^n)$, then $\hat{f} \in L^2(\mathbb{Z}^n)$ and

$$\mathscr{F}: f \mapsto \hat{f}$$

is a *unitary isomorphism* of $L^2(\mathbb{T}^n)$ with $L^2(\mathbb{Z}^n)$:

$$\|f\| = \|\hat{f}\|$$

which is the usual Parseval relation. The inverse operator $\mathscr{F}^{-1}$ is given by

$$f = \sum_m \hat{f}(m)\chi_{-m}$$

the series converging in $L^2(\mathbb{T}^n)$. If we now use $\mathscr{F}$ to carry the representation λ to a representation μ of $\mathbb{T}^n$ in $L^2(\mathbb{Z}^n)$, $\mu = \mathscr{F}\circ\lambda\circ\mathscr{F}^{-1}$, then

$$\mu(\phi)\hat{f}(m) = \chi_m(\phi)\hat{f}(m) \quad (\phi\in\mathbb{T}^n)$$

This formula shows that in the standard basis of $L^2(\mathbb{Z}^n)$ all the operators $\mu(\phi)$ are diagonal.

If f is smooth, $\hat{f}(m)$ tends to 0 very rapidly when $|m|\to\infty$; the series

$$f = \sum_m \hat{f}(m)\chi_{-m}$$

then converges very nicely: we have, for it as well as for all the series obtained by formal differentiation, uniform convergence. In particular,

$$f(0) = \sum_m \hat{f}(m) \quad (f\in C^\infty(\mathbb{T}^n)) \tag{P}$$

We shall refer to this as the *Plancherel formula*.

For $\mathbb{R}^n$ the theory is more delicate. The characters of $\mathbb{R}^n$ are the functions

$$\chi_t: (x_1, \ldots, x_n) \to \exp i(t_1x_1 + \cdots + t_nx_n) \quad (t\in\mathbb{R}^n)$$

The regular representation of $\mathbb{R}^n$ is defined as before; it acts on $L^2(\mathbb{R}^n)$ by

$$(\lambda(y)f)(x) = f(-y+x)$$

Proceeding as before we define the *Fourier transform* of f by

$$\hat{f}(t) = \int_{\mathbb{R}^n} f(x)\chi_t(x)\,\mathrm{d}x \quad (t\in\mathbb{R}^n) \tag{FT}$$

Then

$$(\lambda(y)f)^\wedge(t) = \chi_t(y)\hat{f}(t) \quad (y, t\in\mathbb{R}^n) \tag{M}$$

so that the operators $\lambda(y)$ become multiplication operators simultaneously, and thus are 'diagonalized'. However, the χ_t are of absolute value 1 and so do not lie in the Hilbert space, so that the definition of the Fourier transform in (FT) is not strictly valid for all f in $L^2(\mathbb{R}^n)$. The traditional way to overcome this difficulty is to use the definition (FT) initially for f suitably restricted, say for $f\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$; the key step is then to prove that on this restricted domain the map $f\mapsto\hat{f}$ is essentially unitary, and then to complete its definition to all of $L^2(\mathbb{R}^n)$ by continuity, noting that $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. The restricted unitarity is proved in the form

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(t)|^2 \, dt \qquad (P_1)$$

It then turns out that the Fourier transform maps $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ so that it is a unitary isomorphism:

$$\mathscr{F}: L^2(\mathbb{R}^n, dx) \cong L^2(\mathbb{R}^n, (2\pi)^{-n} \, dt)$$

The relations (M) are now valid rigorously and show that the representation $\mu = \mathscr{F} \circ \lambda \circ \mathscr{F}^{-1}$ acts by multiplication operators. The formula (P_1), valid for all $f \in L^2(\mathbb{R}^n)$, is the *Plancherel formula*.

The 'diagonalization' of λ effected by $\mathscr{F}$ is the classic example of a 'continuous decomposition'. Let us introduce an equivalence relation in the σ-algebra of Borel subsets of $\mathbb{R}^n$ by defining $E \sim F$ to mean $(E \backslash F) \cup (F \backslash E)$ has measure zero. If $\mathscr{B}$ is the set of equivalence classes, $\mathscr{B}$ is a σ-algebra also; but unlike the σ-algebra of Borel sets $\mathscr{B}$ has no atoms. Further, Lebesgue measure becomes a measure on $\mathscr{B}$ with the property that each nonzero element has measure strictly greater than zero. For any Borel set E let

$$S(E) = \{f \mid f \in L^2(\mathbb{R}^n, dx)\}, \hat{f} = 0 \text{ outside } E\}$$

It is then easy to show using (M) that $S(E)$ is a closed λ-stable subspace of $L^2(\mathbb{R}^n, dx)$. Of course $S(E)$ depends only on the equivalence class of E and so we have a map $S(e \mapsto S(e))$ from the σ-algebra $\mathscr{B}$ to the orthocomplemented lattice Λ of the λ-stable closed linear subspaces of $L^2(\mathbb{R}^n, dx)$. It is not difficult to show at this stage that S is an *isomorphism*:

$$S: \mathscr{B} \cong \Lambda$$

The most elegant way to prove all of these assertions is by using the Schwartz space; this method will also bring out the duality explicitly, and will have the additional advantage of focussing on the differential aspects of the theory. The Schwartz space of $\mathbb{R}^n$ is the space $\mathscr{S}$ of all C^∞ functions f on $\mathbb{R}^n$ such that for any integers $m \geqslant 0$, $\alpha_1, \ldots, \alpha_n \geqslant 0$,

$$((\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} f)(x) = O((1 + x_1^2 + \cdots + x_n^2)^{-m})$$

when $x_1^2 + \cdots + x_n^2 \to \infty$. If we introduce the seminorms

$$\mu_{\alpha,m}(f) = \sup |(D^\alpha f)(x)| (1 + x_1^2 + \cdots + x_n^2)^m$$

$(D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n})$, we can view $\mathscr{S}$ as a topological vector space also. It is easy to show that $\mathscr{S}$ is complete (so that $\mathscr{S}$ is a Fréchet space). The differential operators D^α act as continuous linear operators on $\mathscr{S}$. Also, if we write, for any smooth function g, M_g for the operator of multiplication by g, then, under suitable assumptions on g, M_g will be a continuous linear operator on $\mathscr{S}$. For instance this is true if g is a polynomial. More generally, if g is of moderate growth in the sense that for any α there are an integer $m(\alpha) \geqslant 0$ and a constant $C(\alpha) > 0$ such that

$$|(D^\alpha g)(x)| \leqslant C(\alpha)(1 + x_1^2 + \cdots + x_n^2)^{m(\alpha)}$$

for all $x \in \mathbb{R}^n$, then $M_g(f \mapsto gf)$ is a well-defined and continuous operator of $\mathscr{S}$.

The rapid decay of the elements of $\mathscr{S}$ at infinity means that $\mathscr{S} \subset L^1(\mathbb{R}^n)$ and shows at once that the Fourier transform is defined by (FT) for all $f \in \mathscr{S}$; moreover, $\hat{f}$ will be a smooth function of t, and we have the formula

$$(-\mathrm{i}\partial/\partial t_1)^{\beta_1} \cdots (-\mathrm{i}\partial/\partial t_n)^{\beta_n} \hat{f} = (M_{x_1}^{\beta_1} \cdots M_{x_n}^{\beta_n} f)^\wedge$$

(x_j is the coordinate function $(x_1, \ldots, x_n) \mapsto x_j$). Furthermore, replacing f by its derivatives and integrating by parts we find the dual formula

$$((\mathrm{i}\partial/\partial x_1)^{\alpha_1} \cdots (\mathrm{i}\partial/\partial x_n)^{\alpha_n} f)^\wedge = M_{t_1}^{\alpha_1} \cdots M_{t_n}^{\alpha_n} \hat{f}$$

The estimate

$$|\hat{g}(t)| \leqslant \|g\|_1 \quad (g \in \mathscr{S})$$

in conjunction with the above formula now shows that $\hat{f} \in \mathscr{S}$ and that

$$\mathscr{F}: f \mapsto \hat{f}$$

is a continuous linear map of $\mathscr{S}$ into $\mathscr{S}$. The basic theorem may now be formulated as the assertion that $\mathscr{F}$ is a topological linear isomorphism of $\mathscr{S}$ with itself, and that $\mathscr{F}^{-1}$ is given by the inversion formula

$$(\mathscr{F}^{-1}g)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(t)\chi_t(x)\,\mathrm{d}t \qquad \text{(INV)}$$

If we take $x = 0$ and $g = \hat{f}$ in (INV) we get the Plancherel formula in the form

$$f(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(t)\,\mathrm{d}t \quad (f \in \mathscr{S}) \qquad (\mathrm{P}_2)$$

To get the earlier version (P_1) it suffices to take $f = g * \tilde{g}$ where $\tilde{g}(x) = \overline{g(-x)}$, and $*$ is convolution, defined by

$$(h_1 * h_2)(x) = \int_{\mathbb{R}^n} h_1(y)h_2(-y + x)\,\mathrm{d}y \quad (x \in \mathbb{R}^n)$$

for $h_1, h_2 \in \mathscr{S}$; then

$$\hat{f} = |\hat{g}|^2, f(0) = \int_{\mathbb{R}^n} |g(x)|^2\,\mathrm{d}x,$$

and so (P_1) follows from (P_2).

The Plancherel formula (P_2) has an interpretation from the standpoint of Schwartz's theory of distribution [Sch] which is important for us. Let us