Lie Groups and Lie Algebras for Physicists

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To

Mary Okubo

Preface

Group theory was discovered by Évariste Galois in the 19th century for the case of finite dimensional symmetric group. It has been successfully generalized subsequently for the infinite dimensional case (i.e. Lie group and Lie algebra) by Sophus Lie. Before the advent of quantum mechanics in the early 20th century, group theory was thought by many physicists to be unimportant in the study of physics. It is indeed of some interest to note the following anecdote narrated by Freeman Dyson (as quoted in Mathematical Apocrypha, p 21, by S. A. Kranz), "In 1910, the mathematician Oswald Veblen (1880-1960) - a founding member of the Institute for Advanced Study - and the physicist James Jean (1877-1946) were discussing the reform of the mathematics curriculum at Princeton University. Jeans argued that they 'may as well cut out group theory,' for it 'would never be of any use to physics'." The real fundamental change in thinking truly occurred with the development of quantum mechanics. It was soon realized that a deep knowledge of group theory and Lie algebra in the study of of angular momentum algebra is crucial for real understanding of quantum mechanical atomic and nuclear spectral problems. At present, group theory permeates problems in practically every branch of modern physics. Especially in the study of Yang-Mills gauge theory and string theory the use of group theory is essential. For example, we note that the largest exceptional Lie algebra E_8 appears in heterotic string theory and also in some one-dimensional Ising model in statistical mechanics, some of whose predictions have been experimentally verified recently.

Our goal in this monograph is to acquaint (mostly) graduate students of physics with various aspects of modern Lie group and Lie algebra. With this in view, we have kept the presentation of the material in this book at a pedagogical level avoiding unnecessary mathematical rigor. Furthermore, the groups which we will discuss in this book will be mostly Lie groups which are infinite dimensional. We will discuss finite groups only to the extent that they will be necessary for the development of our discussions. The interested readers on this topic are advised to consult many excellent text books on the subject. We have not tried to be exhaustive in the references. Rather, we have given only a few references to books (at the end of each chapter) that will be easier to read for a student with a physics background. We assume that the readers are familiar with the material of at least a two semester graduate course on quantum mechanics as well as with the basics of linear algebra theory.

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Contents

CHAPTER 1

Introduction to groups

In this chapter, we introduce the concept of a group and present in some detail various examples of commonly used groups in physics. This is helpful in establishing the terminology as well as the notations commonly used in the study of groups which will also be useful in further development of various ideas associated with groups.

1.1 Definition of a group

Let us start with the formal definition of a group G as follows:

($G1$): For any two elements a and b in a group, a product is defined in G satisfying

$$ab = c \in G, \quad \forall a, b \in G. \tag{1.1}$$

($G2$): The group product is associative so that

$$(ab)c = a(bc)(\equiv abc), \quad \forall a, b, c \in G. \tag{1.2}$$

($G3$): The group has a unique identity (unit) element $e \in G$ such that

$$ea = ae = a, \quad \forall a \in G. \tag{1.3}$$

This implies that

$$ee = e \in G. \tag{1.4}$$

($G4$): Any element $a \in G$ has a unique inverse element $a^{-1} \in G$ so that

$$aa^{-1} = a^{-1}a = e. \tag{1.5}$$

Any set of elements G satisfying all the axioms $(G1)$-$(G4)$ is defined to be a group. On the other hand, a set of elements which satisfies only the first three axioms $(G1)$-$(G3)$, but not $(G4)$, is known as a semi-group. (More rigorously, a semi-group is defined as the set of elements which satisfy only $(G1)$-$(G2)$. However, one can always add the identity element to the group since its presence, when an inverse is not defined, is inconsequential (see (1.3)-(1.4)) and we will adopt this definition commonly used in physics.)

Some comments are in order here. We note that the definition of a group does not require that the product rule satisfy the commutativity law $ab = ba$. However, if for any two arbitrary elements of the group, $a, b \in G$, the product satisfies $ab = ba$, then the group G is called a commutative group or an Abelian group (named after the Norwegian mathematician Niels Henrik Abel). On the other hand, if the product rule for a group G does not satisfy commutativity law in general, namely, if $ab \neq ba$ for some of the elements $a, b \in G$, then the group G is known as a non-commutative group or a non-Abelian group. Furthermore, it is easy to see from the definition $(G4)$ of the inverse of an element that the inverse of a product of two elements satisfies

$$(ab)^{-1} = b^{-1}a^{-1} \neq a^{-1}b^{-1}, \tag{1.6}$$

in general, unless, of course, the group G is Abelian. Equation (1.6) is easily checked in the following way

$$\begin{aligned}
(ab)(b^{-1}a^{-1}) &= a(b(b^{-1}a^{-1})), && \text{by (1.2) with } c = b^{-1}a^{-1},\\
&= a((bb^{-1})a^{-1}), && \text{by (1.2)},\\
&= a(ea^{-1}), && \text{by (1.5)},\\
&= aa^{-1}, && \text{by (1.3)},\\
&= e, && \text{by (1.5)}.
\end{aligned} \tag{1.7}$$

Similarly, it is straightforward to verify that $(b^{-1}a^{-1})(ab) = e$.

In order to illustrate the proper definitions in a simple manner, let us consider the following practical example from our day to day life. Let "a" and "b" denote respectively the operations of putting on a coat and a shirt. In this case, the (combined) operation "ab" would correspond to putting on a shirt first (b) and then putting on a coat (a) whereas the (combined) operation "ba" would denote putting on a coat first and then a shirt. Clearly, the operations are not commutative, namely, $ab \neq ba$. If we now introduce a third operation "c" as corresponding to putting on an overcoat, then the law of associativity of the operations (1.2) follows, namely, $(ca)b = c(ab) = cab$ and corresponds to putting on a shirt, a coat and an overcoat in that order. It now follows that the operation "bb" denotes putting on two shirts while "$b(bb) = (bb)b = bbb$" stands for putting on three shirts etc. The set of these operations would define a semi-group if we introduce the identity (unit) element e (see (1.3)) to correspond to the operation of doing nothing. However, this does not make the set of operations a group for the following reason. We note that we can naturally define the inverses "a^{-1}" and "b^{-1}" to correspond respectively to the operations of taking off a coat and a shirt. It follows then that $a^{-1}a = e = b^{-1}b$ as required. However, we also note that the operation "aa^{-1}" is not well defined, in general, unless there is already a coat on the body and, therefore, $aa^{-1} \neq e$, in general (but $a^{-1}a = e$ always). As a result, the set of operations so defined do not form a group. Nonetheless, (1.6) continues to be valid, namely, $(ab)^{-1} = b^{-1}a^{-1} \neq a^{-1}b^{-1}$, whenever these operations are well defined. In fact, since ab corresponds to putting on a shirt and then a coat, if we are already dressed in this manner, the inverse $(ab)^{-1}$ would correspond to removing these clothes (come back to the state prior to the operation ab) which can only be done if we first take off the coat and then the shirt. This leads to $(ab)^{-1} = b^{-1}a^{-1}$. Furthermore, let us note that if "d" denotes the operation of putting on a trouser, then the operation of putting on a shirt and a trouser are commutative, namely, $db = bd$ and so on.

1.2 Examples of commonly used groups in physics

In this section, we would discuss some of the most commonly used groups in physics. This would also help to set up the conventional notations associated with such groups.

1.2.1 Symmetric group S_N. Although most of the commonly used groups in physics are infinite dimensional (Lie) groups, let us begin with a finite dimensional group for simplicity. Let us consider a set of N distinct objects labelled by $\{x_i\}, i = 1, 2, \cdots, N$ and consider all possible distinct arrangements or permutations of these elements. As we know, there will be $N!$ possible distinct arrangements (permutations) associated with such a system and all such permutations (operations) form a group known as the symmetric group or the permutation group of N objects and is denoted by S_N. Since the number of elements in the group is finite, such a group is known as a finite group and the number of elements of the group ($n!$) is known as the order of the group S_N. Permutation groups are relevant in some branches of physics as well as in such diverse topics as the study of the Rubik's cube. Note that every twist of the Rubik's cube is a rearrangement of the faces (squares) on the cube and this is how the permutation group enters into the study of this system. There are various possible notations to denote the permutations of the objects in a set, but let us choose the most commonly used notation known as the cycle notation. Here we denote by (ij) the operation of permuting the objects x_i and x_j in the set of the form

$$(ij): \quad x_i \leftrightarrow x_j, \quad i,j = 1,2,\cdots,N, \tag{1.8}$$

with all other objects left unchanged. In fact, the proper way to read the relation (1.8) is as

$$(ij): \quad x_i \rightarrow x_j \rightarrow x_i. \tag{1.9}$$

This brings out the cyclic structure of the cycle notation (parenthesis) and is quite useful when the permutation of more objects are involved. It is clear from this cyclic structure that under a cyclic permutation

$$(ij) = (ji), \tag{1.10}$$

and when permutations of three objects are involved we have

$$(ijk): \quad x_i \rightarrow x_j \rightarrow x_k \rightarrow x_i, \tag{1.11}$$

and

$$(ijk) = (kij) = (jki)\,(\neq (jik) = (kji) = (ikj))\,. \tag{1.12}$$

Let us clarify the group structure of S_N in the case of $N = 3$, namely, let us consider the set of three objects (elements) $\{x_i\} = (x_1, x_2, x_3)$ where $i = 1,2,3$. With the cycle notation we note that the 3! distinct permutations of these elements can be denoted by the set of operations $\{P\}$ with the elements in the set given by

$$\begin{aligned} & P_1 = e = (\mathbb{1}), P_2 = (12), P_3 = (23), P_4 = (13), \\ & P_5 = (123), P_6 = (132), \end{aligned} \tag{1.13}$$

where we have introduced the identity element $P_1 = e = (\mathbb{1})$ to correspond to doing no permutation (every object is left unchanged