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John A. Van de Walle

**Elementary
and
Middle School
Mathematics**

Teaching Developmentally

Third Edition

ELEMENTARY AND MIDDLE SCHOOL MATHEMATICS

TEACHING DEVELOPMENTALLY

THIRD EDITION

JOHN A. VAN DE WALLE

Virginia Commonwealth University



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PREFACE

Children will become confident “doers” of mathematics only if mathematics makes sense to them and if they believe in their ability to make sense of it.

(TRAFTON & CLAUS, 1994, p. 21)

To help students make sense of mathematics and become confident in their ability to do so is an excellent goal for teachers and a principal goal of this book. *Elementary and Middle School Mathematics* is both a guide and an instructional resource to help you with the challenging and rewarding task of helping all children become confident “doers of mathematics.”

The third edition of this book comes nearly a decade after the National Council of Teachers of Mathematics published the *Curriculum and Evaluation Standards for School Mathematics*. That seminal document began an exciting period of change and growth in mathematics education that continues today. We continue to understand more about how children learn mathematics. There is new and exciting content for children to learn. There is new technology to help make it possible. This new edition reflects the ongoing growth and change in mathematics education. It is designed to help you be the most important part of that growth—a facilitator who helps children develop confidence and understanding as they do mathematics.

What You Will Find in This Book

I have divided this book into three sections. Each plays an important role in helping you become an effective teacher. Together they serve as both a resource and a reference book for teaching mathematics.

Section 1: Foundations

The more I work with teachers, the more I find that at all grade levels, a strong understanding of the foundational ideas of mathematics education has the greatest impact on

teacher effectiveness. The best teachers I know have a good sense of how children learn and how to involve them in doing mathematics. In each edition of this book, I have worked hardest at developing and revising these five chapters to make them reflective of current research and practice and to extend the implications of these ideas to the rest of the text. These are not just overview chapters that are required in an academic text. Rather, these are the most important chapters of the book. They provide foundational ideas for the teaching of mathematics at any grade level.

Section 2: Concepts and Procedures

Chapters 6 through 21 build on the foundational ideas found in the first section as each explores the specific mathematics content found in grades K to 8. Every major topic of the K–8 curriculum is addressed in depth. Each chapter provides a perspective on the mathematical content and on how children best learn that content, as well as numerous suggestions for activities to engage children in the process of doing mathematics. This book is unique in its treatment of so many topics in such a thorough manner.

Many teachers use this text as a resource book and consult it as they plan each new unit. It provides them with plenty of activities and a clear understanding of the development of content. The activities for children, each enclosed in a box and identified by a title, are integrated directly into the text. Reflecting on the activities as you read can help you think about the mathematics from the perspective of the child. Most require students to figure something out, to search for a pattern, to explain a reason for an idea—to do mathematics as problem solving. As a teacher, you are encouraged to go beyond students’ answers and discuss the results of activities, requiring them to justify and explain their results or to elaborate new ideas.

The activities are also there for you. They are meant to be read along with the text, not as asides. After all, you are constructing new knowledge yourself—knowledge about teaching mathematics to children. Like your students, you must be actively engaged in *your own learning* about *children learning* mathematics. By actually doing the activities as you

read through the book, you can get an idea of how children might react to or learn from each activity.

Section 3: Issues and Perspectives

That these three chapters are found at the end of the book should in no way diminish their importance. As the foundational chapters of Section 1 provide a framework for the teaching of mathematics, the ideas in this section offer perspective on overarching issues that also build on that foundation. Read these chapters, or portions of them, at any time, depending on your needs and interests.

How do you design a lesson? In Chapter 22, two general approaches are offered so that you can develop your own style and still use a constructivist approach. How do you form cooperative learning groups? Details based on expert thinking are provided. Answers or suggestions are also offered for working in a diverse classroom, drill and practice, writing in mathematics, homework, and the use of the traditional textbook.

NCTM has made it very clear that good mathematics should be accessible to all children. In Chapter 23, you will read about working with children with learning disabilities, children with mental disabilities, and gifted children. In addition, an up-to-date perspective is provided on multicultural and socioeconomic issues. Gender equity is also discussed. Though no easy answers are offered, the information is current, research-based, and thought-provoking, in keeping with the main ideas of the book.

Chapter 24 looks at the role of the calculator and the computer in teaching mathematics. Here you will find NCTM's position on the use of calculators and computers and a framework that will guide your use and selection of technology as you teach. Included are ideas for profitable use of the Internet and a look at the Logo computer programming language.

Special Features of This Text

This third edition of *Elementary and Middle School Mathematics* retains all of the features that I believe have helped make the previous two editions so successful. A few new additions have been included to make it even better. Here are 11 things to look for.

Big Ideas (New!)

A constructivist perspective calls teachers to plan activities around big ideas rather than tiny skills and concepts. To help develop an understanding of what is meant by “big ideas,” each chapter in Section 2 begins with a brief overview and a listing of the big conceptual ideas related to that chapter. As you read these chapters, the Big Ideas list will help you begin to see how smaller ideas are actually developed as a part of these larger concepts.

Activities

This is both a textbook and a resource for teachers. I have included as many activities as space will allow. Most of these are clearly identified by a box and a title. Other ideas are described directly in the text or in the illustrations. Every activity found here should be considered for its potential for doing mathematics as problem solving, as described in Chapter 4. You are encouraged to use and modify activities in ways that challenge children to think and to wrestle with ideas.

Drawings

You will find no decorative or unnecessary art in this text. Every drawing is an integral part of the text information and should not be overlooked. Color is used within the illustrations to make them even more explanatory. Drawings are used whenever a picture seemed better than words or to highlight the most important ideas. I frequently tell my students to “read the pictures” as one of the fastest ways to the important information in the book.

Assessment Notes (New!)

Near the end of most chapters in Section 2, you will find a short discussion headed “Assessment Notes.” These sections are designed to point to special assessment ideas that build on the general themes of the chapter on assessment, Chapter 5. It is worth noting that most activities in the book are also appropriate as performance assessments. There is no need to have a special reference book for assessment tasks.

Literature Connections (New!)

Children's literature is one of the most inviting ways to get children actively involved in doing mathematics. Many of the chapters in Section 2 contain a feature titled “Literature Connections.” Generally you will find at least three children's literature titles suggested, with a brief description of how the mathematics of the chapter can be profitably built on the stories. Though certainly not a comprehensive listing of potential literature, these annotated listings provide all you need to get started using this exciting vehicle for teaching mathematics. Each work featured under this head appears in a separate reference list at the back of the book.

Computers and Calculators

Calculator activities are found in every chapter involving numbers and are identified with a small calculator icon so that you can easily find them. Special efforts have been made in this third edition to expand the use of graphing calculators for middle school activities. Computer activities are included in instances when the computer can have a unique impact on learning. Chapter 24 expands on the use of technology even further.

Writing to Learn

To help you focus on the important pedagogical ideas, a list of focusing questions are found at the end of every chapter under the heading “Reflections on Chapter N: Writing to Learn.” These questions are designed to help you reflect on the main points of the chapter. Actually writing out the answers to these questions in your own words is one of the best ways for you to develop your understanding of each chapter’s main ideas.

Explorations

Following the Writing to Learn list are a few additional questions that ask you to explore an issue, reflect on observations in a classroom, compare text ideas with those found in traditional curriculum materials, or perhaps take a position on a controversial issue. There are rarely “right” answers to these questions, but they will certainly stimulate thought and perhaps even provoke spirited conversations. Discuss these with your peer group or with teachers in the classroom.

Suggested Readings

The end of each chapter contains a bibliography of useful reading selections to augment the information found in the chapter. Usually these are taken from NCTM journals, yearbooks, and other professional resources that are targeted for the classroom teacher. New in this edition, three or four of the selections deemed most important or most useful are provided with a short annotation under the head “Highly Recommended.” It would have been nice to annotate the whole list but space does not permit. (Note that all sources cited within the text proper appear in the References at the back of the book.)

NCTM Standards (New!)

This edition contains listings of standards taken directly from each of the NCTM’s three *Standards* documents. In Appendix A, you will find all of the standards from the K–4 and 5–8 sections of the *Curriculum and Evaluation Standards for School Mathematics*. As in the previous editions, the tables of increased and decreased attention from these two sections are also included. Appendix B contains the six Standards for Teaching Mathematics from the *Professional Standards for Teaching Mathematics*. In Chapter 5, you will find the six Assessment Standards as well as the four Purposes of Assessment found in the *Assessment Standards for School Mathematics*.

Blackline Masters

The Blackline Masters section at the end of the book offers an extensive collection of masters and also directions for making important instructional materials. Many of these masters are unique to this book and support especially interesting activities. Suggestions for the use of these materials

are found throughout the book. You are encouraged to copy these pages and duplicate them for your classroom or for activities with children. (Permission to copy these pages is granted on the copyright page.)

New in This Edition

Revisions for the third edition went far beyond the addition of the new features or a change in design. I have made a serious effort to make this text reflect the most current thinking about teaching mathematics that I thought was possible and still be a useful text with which a preservice teacher can grow. Perhaps as much as 40 percent of the text is new, and revisions have been made throughout every chapter. Though the basic structure and philosophy of the book remain the same, I have worked to improve this edition in very specific ways.

New Foundations Chapters

Section 1 parallels the first five chapters of the second edition, but I believe these chapters show significant improvement and, with respect to problem solving, a significant change.

All five chapters have been completely rewritten. Chapters 1, 2, and 5 are better organized, with a more focused purpose. Let me comment a bit more on Chapters 3 and 4.

Chapter 3, “Developing Understanding in Mathematics,” is more tightly focused on the constructivist view of learning. This rewritten and reorganized chapter reflects my own professional growth as a constructivist. I have continued to learn from teachers and from many things that have been written at the teacher level about how children learn. You will find a much expanded discussion of the role that models play in learning. It is important for teachers to understand that the presence of manipulatives in no way guarantees learning. The chapter is now more readable, more focused on children, and I hope more friendly while remaining challenging.

Chapter 4, “Teaching Through Problem Solving,” is completely new and unique among texts of this kind. Here I have taken to heart the first of the *Curriculum Standards*, mathematics as problem solving. This chapter now describes a primary approach to the teaching of all mathematics. The thesis developed is that all mathematics can be taught in a problem-solving environment. Understanding what that means can and should guide the design of lessons and interaction in the classroom. Problem solving is, then, no longer a separate strand of the curriculum but a way of learning and doing mathematics.

Changes in the Content Chapters

The most apparent change in Section 2 is the addition of Chapter 21, “Functions and Variables.” This is a completely

new chapter in which function concepts are developed through five modes: real contexts, language, equations, tables, and graphs. The chapter leans heavily on the use of the graphing calculator, builds on the development of functions through data and through patterns, and avoids the tedium of formalistic approaches. With the addition of this chapter and modifications to other content for the middle grades, the text now completely covers all important content in grades 6 to 8.

Other notable changes can be found in the chapters on operations, place value, and computation. However, no chapter is completely without revision. I made a serious effort to reflect on every chapter in the book. Every activity was reconsidered to be certain that it could be incorporated profitably into the problem-solving view developed in Chapter 4. Many were modified or replaced.

New Issues and Perspective

All three chapters in Section 3 are completely new. The content of these chapters was described earlier. I am proud to say that these chapters represent the most current thinking in the profession. Discussions of diversity in the classroom, writing in mathematics, and drill and practice are completely new. There is enough detail in the planning chapter to make it useful and effective for a teacher wishing to implement the ideas of this book.

In Chapter 23, “Teaching All Children Mathematics,” the notion that all children can learn mathematics in much the same manner is supported by research and by examples of successful projects. New topics include multicultural issues and gender equity.

The technology chapter is designed to offer perspective on the use of calculators and computers, reflecting recent advances in these areas. It includes the most recent position statement on technology from NCTM. A new section on the Internet is designed to introduce teachers to this resource for mathematics education and teaching ideas.

Notes to the Instructor

To my knowledge, no other text offers your students as much as this one does. Most important, it develops a strong theoretical perspective of children learning mathematics—not a casual overview. That perspective is reflected throughout the book. Second, prospective teachers find the book to be a valuable resource that almost all carry into the classroom rather than sell back to the bookstore. Many classroom teachers and school principals buy the book for the same purpose. I have tried to write in a style that respects teachers as professionals without asking them to wade through educational jargon. My students tell me it is a readable and usable book.

It remains a very long book, full of more ideas and information than any other text. It is not a book to “cover” but a

book to use. I have never attempted to teach the content of the entire book in a single semester and would never expect you to. In the past, I have seen the content coverage and detail as a luxury rather than a burden. You are able to pick and choose topics you like to emphasize, have some topics assigned as reading, and design a course that best suits your purposes.

With this edition, I have a different thought to add to these options. I now believe that the five chapters in Section 1 provide the core material essential to becoming an effective teacher. A possible suggestion is to spend a significant portion of your course on developing these key ideas about teaching children. Use an even smaller list of content topics than you would have before. Let the content you select be a vehicle for discussion of the general themes. Many chapters can be read without your input. Even those you do not assign will most likely be taken into the classroom for future use. By developing the foundational ideas thoroughly and placing a quality resource in your students’ hands, you will, I believe, achieve even more than trying to cover every topic in the elementary or middle grades.

As I continue to talk to friends around the country who have used the previous editions, I am struck by the many different styles of methods courses that are possible. All instructors seem to find their own best way to use the book. I am hesitant, therefore, to dictate a course, but I am proud to offer this text. I wish you and your teachers much success and excitement as you explore good mathematics.

ACKNOWLEDGMENTS

Though much revised in this edition and the second, the general approach of *Elementary and Middle School Mathematics* was developed with the first edition. Substantial credit for the success of the first edition belongs to the mathematics educators who gave time and great care in offering detailed comments on the original draft manuscript. Regardless of how many subsequent editions this text may see, I will always be most sincerely indebted to John Dossey of Illinois State University, Bob Gilbert of Florida International University, Warren Crown of Rutgers University, and Steven Willoughby of the University of Arizona. Few mathematics educators of their stature would take the time and effort that they gave to this endeavor.

In preparing this third edition, I have received thoughtful input from the following educators, who offered comments on the second edition and/or on the manuscript for the third:

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A Personal Note

Working on this book over the years has often been a lonely, daunting, and humbling endeavor. The extraordinary support, encouragement, and patience that my wife, Sharon, has given to me without reservation or complaint has made working on the first and subsequent editions possible. Her faith in me has helped me believe in myself. Her enduring love has made it all worthwhile. She has spent many long nights and weekends alone so that I could work on "the book." In this endeavor and throughout our lives together, she has been my strongest supporter. I have watched as she has grown in her own professional life and treasure her support of mine all the more.

With all my love, thank you, Sharon.

JOHN A. VAN DE WALLE

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