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Toric Topology

Introduction

Traditionally, the study of torus actions on topological spaces has been considered as a classical field of algebraic topology. Specific problems connected with torus actions arise in different areas of mathematics and mathematical physics, which results in permanent interest in the theory, new applications and penetration of new ideas into topology.

Since the 1970s, algebraic and symplectic viewpoints on torus actions have enriched the subject with new combinatorial ideas and methods, largely based on the convex-geometric concept of polytopes.

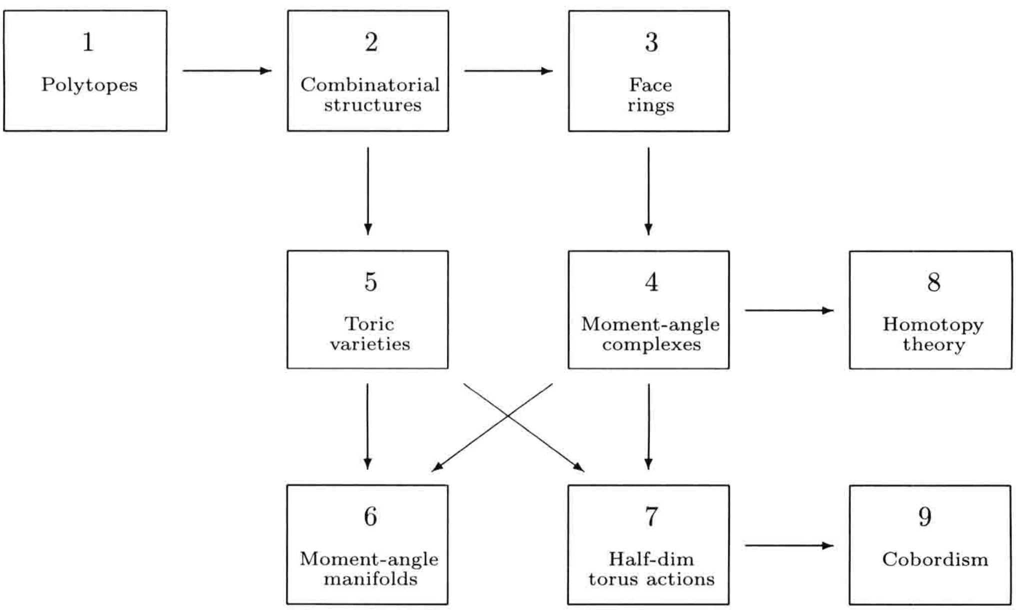
The study of algebraic torus actions on algebraic varieties has quickly developed into a highly successful branch of algebraic geometry, known as *toric geometry*. It gives a bijection between, on the one hand, *toric varieties*, which are complex algebraic varieties equipped with an action of an algebraic torus with a dense orbit, and on the other hand, *fans*, which are combinatorial objects. The fan allows one to completely translate various algebraic-geometric notions into combinatorics. Projective toric varieties correspond to fans which arise from convex polytopes. A valuable aspect of this theory is that it provides many explicit examples of algebraic varieties, leading to applications in deep subjects such as singularity theory and mirror symmetry.

In symplectic geometry, since the early 1980s there has been much activity in the field of Hamiltonian group actions on symplectic manifolds. Such an action defines the *moment map* from the manifold to a Euclidean space (more precisely, the dual Lie algebra of the torus) whose image is a convex polytope. If the torus has half the dimension of the manifold, the image of the moment map determines the manifold up to equivariant symplectomorphism. The class of polytopes which arise as the images of moment maps can be described explicitly, together with an effective procedure for recovering a symplectic manifold from such a polytope. In symplectic geometry, as in algebraic geometry, one translates various geometric constructions into the language of convex polytopes and combinatorics.

There is a tight relationship between the algebraic and the symplectic pictures: a projective embedding of a toric manifold determines a symplectic form and a moment map. The image of the moment map is a convex polytope that is dual to the fan. In both the smooth algebraic-geometric and the symplectic situations, the compact torus action is locally isomorphic to the standard action of $(S^1)^n$ on $\mathbb{C}^n$ by rotation of the coordinates. Thus the quotient of the manifold by this action is naturally a manifold with corners, stratified according to the dimension of the stabilisers, and each stratum can be equipped with data that encodes the isotropy torus action along that stratum. Not only does this structure of the quotient provide a powerful means of investigating the action, but some of its subtler combinatorial properties may also be illuminated by a careful study of the equivariant topology of the manifold. Thus, it should come as no surprise that since the beginning of the 1990s, the ideas and methodology of toric varieties and Hamiltonian torus actions have started penetrating back into algebraic topology.

By 2000, several constructions of topological analogues of toric varieties and symplectic toric manifolds had appeared in the literature, together with different seemingly unrelated realisations of what later has become known as moment-angle manifolds. We tried to systematise both known and emerging links between torus actions and combinatorics in our 2000 paper [**67**] in Russian Mathematical Surveys, where the terms 'moment-angle manifold' and 'moment-angle complex' first appeared. Two years later it grew into a book *Torus Actions and Their Applications in Topology and Combinatorics* [**68**] published by the AMS in 2002 (the extended Russian edition [**69**] appeared in 2004). The title 'Toric Topology' coined by our colleague Nigel Ray became official after the 2006 Osaka conference under the same name. Its proceedings volume [**177**] contained many important contributions to the subject, as well as the introductory survey *An Invitation to Toric Topology: Vertex Four of a Remarkable Tetrahedron* by Buchstaber and Ray. The vertices of the 'toric tetrahedron' are topology, combinatorics, algebraic and symplectic geometry, and they have symbolised many strong links between these subjects. With many young researchers entering the subject and conferences held around the world every year, toric topology has definitely grown into a mature area. Its various aspects are presented in this monograph, with an intention to consolidate the foundations and stimulate further applications.

Chapter guide



Each chapter and most sections have their own introductions with more specific information about the contents. 'Additional topics' of Chapters 1, 3 and 4 contain more specific material which is not used in an essential way in the following chapters. The appendices at the end of the book contain material of more general nature, not exclusively related to toric topology. A more experienced reader may refer to them only for notation and terminology.

At the heart of toric topology lies a class of torus actions whose orbit spaces are highly structured in combinatorial terms, that is, have lots of orbit types tied together in a nice combinatorial way. We use the generic terms *toric space* and *toric object* to refer to a topological space with a nice torus action, or to a space produced from a torus action via different standard topological or categorical constructions. Examples of toric spaces include toric varieties, toric and quasitoric manifolds and their generalisations, moment-angle manifolds, moment-angle complexes and their Borel constructions, polyhedral products, complements of coordinate subspace arrangements, intersections of real or Hermitian quadrics, etc.

In Chapter 1 we collect background material related to convex polytopes, including basic convex-geometric constructions and the combinatorial theory of face vectors. The famous g-theorem describing integer sequences that can be the face vectors of simple (or simplicial) polytopes is one of the most striking applications of toric geometry to combinatorics. The concepts of Gale duality and Gale diagrams are important tools for the study of moment-angle manifolds via intersections of quadrics. In the additional sections we describe several combinatorial constructions providing families of simple polytopes, including nestohedra, graph associahedra, flagtopes and truncated cubes. The classical series of permutahedra and associahedra (Stasheff polytopes) are particular examples. The construction of nestohedra takes its origin in singularity and representation theory. We develop a differential algebraic formalism which links the generating series of nestohedra to classical partial differential equations. The potential of truncated cubes in toric topology is yet to be fully exploited, as they provide an immense source of explicitly constructed toric spaces.

In Chapter 2 we describe systematically combinatorial structures that appear in the orbit spaces of toric objects. Besides convex polytopes, these include fans, simplicial and cubical complexes, and simplicial posets. All these structures are objects of independent interest for combinatorialists, and we emphasised the aspects of their combinatorial theory most relevant to subsequent topological applications.

The subject of Chapter 3 is the algebraic theory of face rings (also known as Stanley–Reisner rings) of simplicial complexes, and their generalisations to simplicial posets. With the appearance of face rings at the beginning of the 1970s in the work of Reisner and Stanley, many combinatorial problems were translated into the language of commutative algebra, which paved the way for their solution using the extensive machinery of algebraic and homological methods. Algebraic tools used for attacking combinatorial problems include regular sequences, Cohen–Macaulay and Gorenstein rings, Tor-algebras, local cohomology, etc. A whole new thriving field appeared on the borders of combinatorics and algebra, which has since become known as *combinatorial commutative algebra*.

Chapter 4 is the first 'toric' chapter of the book; it links the combinatorial and algebraic constructions of the previous chapters to the world of toric spaces. The concept of the moment-angle complex $\mathcal{Z}_\mathcal{K}$ is introduced as a functor from the category of simplicial complexes $\mathcal{K}$ to the category of topological spaces with torus actions and equivariant maps. When $\mathcal{K}$ is a triangulated manifold, the moment-angle complex $\mathcal{Z}_\mathcal{K}$ contains a free orbit $\mathcal{Z}_\varnothing$ consisting of singular points. Removing this orbit we obtain an open manifold $\mathcal{Z}_\mathcal{K}\setminus\mathcal{Z}_\varnothing$, which satisfies the relative version of

Poincaré duality. Combinatorial invariants of simplicial complexes $\mathcal{K}$ therefore can be described in terms of topological characteristics of the corresponding moment-angle complexes $\mathcal{Z}_\mathcal{K}$. In particular, the face numbers of $\mathcal{K}$, as well as the more subtle *bigraded Betti numbers* of the face ring $\mathbb{Z}[\mathcal{K}]$ can be expressed in terms of the cellular cohomology groups of $\mathcal{Z}_\mathcal{K}$. The integral cohomology ring $H^*(\mathcal{Z}_\mathcal{K})$ is shown to be isomorphic to the Tor-algebra $\mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_m]}(\mathbb{Z}[\mathcal{K}],\mathbb{Z})$. The proof builds upon a construction of a ring model for *cellular* cochains of $\mathcal{Z}_\mathcal{K}$ and the corresponding cellular diagonal approximation, which is functorial with respect to maps of moment-angle complexes induced by simplicial maps of $\mathcal{K}$. This functorial property of the cellular diagonal approximation for $\mathcal{Z}_\mathcal{K}$ is quite special, due to the lack of such a construction for general cell complexes. Another result of Chapter 4 is a homotopy equivalence (an equivariant deformation retraction) from the complement $U(\mathcal{K})$ of the arrangement of coordinate subspaces in $\mathbb{C}^m$ determined by $\mathcal{K}$ to the moment-angle complex $\mathcal{Z}_\mathcal{K}$. Particular cases of this result are known in toric geometry and geometric invariant theory. It opens a new perspective on moment-angle complexes, linking them to the theory of configuration spaces and arrangements.

Toric varieties are the subject of Chapter 5. This is an extensive area with a vast literature. We outline the influence of toric geometry on the emergence of toric topology and emphasise combinatorial, topological and symplectic aspects of toric varieties. The construction of moment-angle manifolds via nondegenerate intersections of Hermitian quadrics in $\mathbb{C}^m$, motivated by symplectic geometry, is also discussed here. Some basic knowledge of algebraic geometry may be required in Chapter 5. Appropriate references are given in the introduction to the chapter.

The material of the first five chapters of the book should be accessible for a graduate student, or a reader with a very basic knowledge of algebra and topology. These five chapters may be also used for advanced courses on the relevant aspects of topology, algebraic geometry and combinatorial algebra. The general algebraic and topological constructions required here are collected in Appendices A and B respectively. The last four chapters are more research-oriented.

Geometry of moment-angle manifolds is studied in Chapter 6. The construction of moment-angle manifolds as the level sets of toric moment maps is taken as the starting point for the systematic study of intersections of Hermitian quadrics via Gale duality. Following a remarkable discovery by Bosio and Meersseman of complex-analytic structures on moment-angle manifolds corresponding to simple polytopes, we proceed by showing that moment-angle manifolds corresponding to a more general class of complete simplicial fans can also be endowed with complex-analytic structures. The resulting family of *non-Kähler* complex manifolds includes the classical series of Hopf and Calabi–Eckmann manifolds. We also describe important invariants of these complex structures, such as the Hodge numbers and Dolbeault cohomology rings, study holomorphic torus principal bundles over toric varieties, and establish collapse results for the relevant spectral sequences. We conclude by exploring the construction of A. E. Mironov providing a vast family of Lagrangian submanifolds with special minimality properties in complex space, complex projective space and other toric varieties. Like many other geometric constructions in this chapter, it builds upon the realisation of the moment-angle manifold as an intersection of quadrics.

In Chapter 7 we discuss several topological constructions of even-dimensional manifolds with an effective action of a torus of half the dimension of the manifold. They can be viewed as topological analogues and generalisations of compact nonsingular toric varieties (or *toric manifolds*). These include *quasitoric manifolds* of Davis and Januszkiewicz, *torus manifolds* of Hattori and Masuda, and *topological toric manifolds* of Ishida, Fukukawa and Masuda. For all these classes of toric objects, the equivariant topology of the action and the combinatorics of the orbit spaces interact in a harmonious way, leading to a host of results linking topology with combinatorics. We also discuss the relationship with GKM-manifolds (named after Goresky, Kottwitz and MacPherson), another class of toric objects having its origin in symplectic topology.

Homotopy-theoretical aspects of toric topology are the subject of Chapter 8. This is now a very active area. Homotopy techniques brought to bear on the study of polyhedral products and other toric spaces include model categories, homotopy limits and colimits, higher Whitehead and Samelson products. The required information about categorical methods in topology is collected in Appendix C.

In Chapter 9 we review applications of toric methods in a classical field of algebraic topology, complex cobordism. It is a generalised cohomology theory that combines both geometric intuition and elaborate algebraic techniques. The toric viewpoint brings an entirely new perspective on complex cobordism theory in both its nonequivariant and equivariant versions.

The later chapters require more specific knowledge of algebraic topology, such as characteristic classes and spectral sequences, for which we recommend respectively the classical book of Milnor and Stasheff [**273**] and the excellent guide by McCleary [**260**]. Basic facts and constructions from bordism and cobordism theory are given in Appendix D, while the related techniques of formal group laws and multiplicative genera are reviewed in Appendix E.

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Contents

CHAPTER 1

Geometry and Combinatorics of Polytopes

This chapter is an introductory survey of the geometric and combinatorial theory of convex polytopes, with the emphasis on those of its aspects related to the topological applications later in the book. We do not assume any specific knowledge of the reader here. Algebraic definitions (graded rings and algebras) required at the end of this chapter are contained in Section A.1 of the Appendix.

Convex polytopes have been studied since ancient times. Nowadays both combinatorial and geometrical aspects of polytopes are presented in numerous textbooks and monographs. Among them are the classical monograph [**166**] by Grünbaum and Ziegler's more recent lectures [**369**]. Face vectors and other combinatorial topics are discussed in books by McMullen–Shephard [**266**], Brøndsted [**49**], and the survey article [**220**] by Klee and Kleinschmidt; while Yemelichev–Kovalev–Kravtsov [**362**] focus on applications to linear programming and optimisation. All these sources may be recommended for the subsequent study of the theory of polytopes, and contain a host of further references.

1.1. Convex polytopes

Definitions and basic constructions. Let $\mathbb{R}^n$ be n-dimensional Euclidean space with the scalar product $\langle\ ,\ \rangle$. There are two constructively different ways to define a convex polytope in $\mathbb{R}^n$:

Definition 1.1.1. A *convex polytope* is the convex hull $\operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q)$ of a finite set of points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q \in \mathbb{R}^n$.

Definition 1.1.2. A *convex polyhedron* P is a nonempty intersection of finitely many half-spaces in some $\mathbb{R}^n$:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0 \quad \text{for } i = 1, \ldots, m\}, \tag{1.1}$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. A *convex polytope* is a bounded convex polyhedron.

All polytopes in this book will be convex. The two definitions above produce the same geometrical object, i.e. a subset of $\mathbb{R}^n$ is the convex hull of a finite point set if and only if it is a bounded intersection of finitely many half-spaces. This classical fact is proved in many textbooks on polytopes and convex geometry, and it lies at the heart of many applications of polytope theory to linear programming and optimisation, see e.g. [**369**, Theorem 1.1].

The *dimension* of a polyhedron is the dimension of its affine hull. We often abbreviate a 'polyhedron of dimension n' to n-*polyhedron*. A *supporting hyperplane* of P is an affine hyperplane H which has common points with P and for which the polyhedron is contained in one of the two closed half-spaces determined by the hyperplane. The intersection $P \cap H$ with a supporting hyperplane is called a *face* of the polyhedron. Denote by ∂P and $\operatorname{int} P = P \setminus \partial P$ the topological boundary and interior of P respectively. In the case $\dim P = n$ the boundary ∂P is the union of all faces of P. Each face of an n-polyhedron (n-polytope) is itself a polyhedron (polytope) of dimension $< n$. Zero-dimensional faces are called *vertices*, one-dimensional faces are *edges*, and faces of codimension one are *facets*.

Two polytopes $P \subset \mathbb{R}^{n_1}$ and $Q \subset \mathbb{R}^{n_2}$ of the same dimension are said to be *affinely equivalent* (or *affinely isomorphic*) if there is an affine map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ establishing a bijection between the points of the two polytopes. Two polytopes are *combinatorially equivalent* if there is a bijection between their faces preserving the inclusion relation. Note that two affinely isomorphic polytopes are combinatorially equivalent, but the opposite is not true.

The faces of a given polytope P form a partially ordered set (a *poset*) with respect to inclusion. It is called the *face poset* of P. Two polytopes are combinatorially equivalent if and only if their face posets are isomorphic.

DEFINITION 1.1.3. A *combinatorial polytope* is a class of combinatorially equivalent polytopes.

Many topological constructions later in this book will depend only on the combinatorial equivalence class of a polytope. Nevertheless, it is always helpful, and sometimes necessary, to keep in mind a particular geometric representative P rather than thinking in terms of abstract posets. Depending on the context, we shall denote by P, Q, etc., geometric polytopes or their combinatorial equivalent classes (combinatorial polytopes). Whenever we consider both geometric and combinatorial polytopes, we shall use the notation $P \approx Q$ for combinatorial equivalence.

We refer to (1.1) as a *presentation* of the polyhedron P by inequalities. These inequalities contain more information than the polyhedron P, for the following reason. It may happen that some of the inequalities $\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle + b_i \geqslant 0$ can be removed from the presentation without changing P; we refer to such inequalities as *redundant*. A presentation without redundant inequalities is called *irredundant*. An irredundant presentation of a given polyhedron is unique up to multiplication of pairs $(\boldsymbol{a}_i, b_i)$ by positive numbers.

EXAMPLE 1.1.4 (simplex and cube). An n-dimensional *simplex* Δ^n is the convex hull of $n+1$ points in $\mathbb{R}^n$ that do not lie on a common affine hyperplane. All faces of an n-simplex are simplices of dimension $< n$. Any two n-simplices are affinely equivalent. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard basis in $\mathbb{R}^n$. The n-simplex $\operatorname{conv}(\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ is called *standard*. Equivalently, the standard n-simplex is specified by the $n+1$ inequalities

$$x_i \geqslant 0 \quad \text{for } i = 1, \ldots, n, \quad \text{and} \quad -x_1 - \cdots - x_n + 1 \geqslant 0. \tag{1.2}$$

The *regular* n-simplex is the convex hull of the endpoints of $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$ in $\mathbb{R}^{n+1}$.

The *standard n-cube* is given by

$$\mathbb{I}^n = [0,1]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon 0 \leqslant x_i \leqslant 1 \quad \text{for } i = 1, \ldots, n\}. \tag{1.3}$$

Equivalently, the standard n-cube is the convex hull of 2^n points $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n$, where $\varepsilon_i = 0$ or 1.Whenever we work with combinatorial polytopes, we shall refer to any polytope combinatorially equivalent to $\mathbb{I}^n$ as a *cube*, and denote it by I^n.

The cube $\mathbb{I}^n$ has $2n$ facets. We denote by F_k^0 the facet specified by the equation $x_k = 0$, and by F_k^1 that specified by the equation $x_k = 1$, for $1 \leqslant k \leqslant n$.