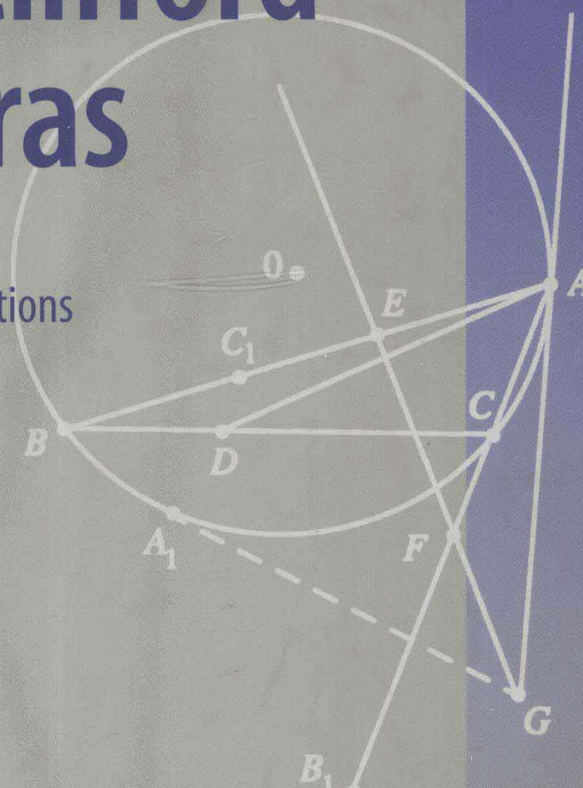
G. SOMMER (ED.)

Geometric Computing with Clifford Algebras

Theoretical Foundations and Applications in Computer Vision and Robotics



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With 89 Figures and 16 Tables

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Editor

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Preface

This book presents a collection of contributions concerning the task of solving geometry related problems with suitable algebraic embeddings. It is not only directed at scientists who already discovered the power of Clifford algebras for their field, but also at those scientists who are interested in Clifford algebras and want to see how these can be applied to problems in computer science, signal theory, neural computation, computer vision and robotics. It was therefore tried to keep this book accessible to newcomers to applications of Clifford algebra while still presenting up to date research and new developments.

The aim of the book is twofold. It should contribute to shift the fundamental importance of adequate geometric concepts into the focus of attention, but also show the algebraic aspects of formulating a particular problem in terms of Clifford algebra. Using such an universal, general and powerful algebraic frame as Clifford algebra, results in multiple gains, such as completeness, linearity and low symbolic complexity of representations. Even problems which may not usually be classified as geometric, might be better understood by the human mind when expressed in a geometric language.

As a misleading tendency, mathematical education with respect to geometric concepts disappears more and more from curricula of technical subjects. To a certain degree this is caused by the mathematicians themselves. What mathematicians today understand as geometry or algebraic geometry is far from beeing accessible to engineers or computer scientists. This is the more regrettable as the Erlangen program of Felix Klein [FK95] on the strong relations between algebra and geometry is of great potential also for the applied sciences.

This book is a first attempt to overcome this situation. As computer scientists and engineers know in principle of the importance of algebra to gain new qualities of modelling, they will profit from geometric interpretations of their models. This was also the experience the authors of this book had made. However, it is not necessarily trivial to translate a geometry related problem into the language of Clifford algebra. Once translated, it also needs some experience to manipulate Clifford algebra expressions. The many applied problems presented in this book should give engineers, computer scientists and physicists a rich set of examples of how to work with Clifford algebra.

The term 'geometric problem' will at times be understood very loosely. For instance, what relation exists between a Fourier transform or its computation as FFT (fast Fourier transform algorithm) and geometry? It will become clear that the Fourier transform is strongly related to symmetry as geometric entity and that its multidimensional extension necessitates the use of an adequate algebraic frame if all multidimensional symmetries are to be kept accessible.

William K. Clifford (1845–1879) [47] introduced what he called "geometric algebra[1]". It is a generalisation of Hermann G. Grassmann's (1809–1877) exterior algebra and also contains William R. Hamilton's (1805–1865) quaternions. Geometric or Clifford algebra has therefore a strong unifying aspect, since it allows us to view different geometry related algebraic systems as specializations of one "mother algebra". Clifford algebras may therefore find a use in quite different fields of science, while still sharing the same fundamental properties.

David Hestenes was one of the first who revived GA in the mid 1960's and introduced it in different fields of physics with the goal to make it a unified mathematical language that encompasses the system of complex numbers, the quaternions, Grassmann's exterior algebra, matrix algebra, vector, tensor and spinor algebras and the algebra of differential forms. In order to achieve this, he fashioned his GA as a specialization of the general CA which is particularly well suited for the use in physics and, as it turned out, engineering and computer science. It is his merit that GA got widely accepted in diverse fields of physics [112, 109] and engineering. The algebra most similar to Hestenes's GA is probably Grassmann's exterior algebra which is also used by a large part of the physics community [FR97]. Those readers who are interested in the evolution of the relations between algebra and geometry can find a short overview in [YAG88].

I first became interested in Clifford or geometric algebra by reading a short paper in Physics World, written by Anthony Garrett [AG92]. At that time I was searching for a way to overcome some serious problems of complete representations of local, intrinsically multidimensional structures in image processing. I got immediately convinced that the algebraic language presented in this paper would open the door to formulate a real multidimensional and linear signal theory. Since then we not only learned how to proceed in that way, but we also discovered the expressive power of GA for quite different aspects of multidimensional signal structure [34, FSR2000].

In the Cognitive Systems research group in Kiel, Germany, we are working on all aspects concerning the design of seeing robot systems [SO99]. This includes pattern recognition with neural networks, computer vision, multidimensional signal theory and robot kinematics. We found that in all these

[1] Today the terms "geometric algebra" (GA) and "Clifford algebra" (CA) are being used interchangeably.

fields GA is a particularly useful algebraic frame. In the process of this research we made valuable experiences of how to model problems with GA. Several contributions to this book present this work[2].

Designing a seeing robot system is a task where quite a number of different competences have to be modelled mathematically. However, in the end the whole system should appear as one. Furthermore, all competences have to be organized or have to organize themselves in a cycle, which has perception and action as two poles. Therefore, it is important to have a common mathematical language to bring the diverse mathematical disciplines, contributing to the diverse aspects of the perception-action cycle, closer together and eventually to fuse them to a general conception of behaviour based system design. In 1997 we brought to life the international workshop on algebraic frames for the perception-action cycle (AFPAC) [217], with the intention to further this fusion of disciplines under the umbrella of a unified algebraic frame. This workshop brought together researchers from all over the world, many of whom became authors in this book. In this respect this book may be considered a collection of research results inspired by the AFPAC'97. Hopefully, the AFPAC 2000 workshop will be of comparable success.

Another mid-range goal is the design of GA processors for real-time computations in robot vision. Today we have to accept a great gap between the low symbolic complexity on the one hand and the high numeric complexity of coding in GA on the other hand. Because available computers cannot even process complex numbers directly, we have to pay a high computational cost at times, when using GA libraries. In some cases this is already compensated by the gain achieved through a concise problem formulation with GA. Nevertheless, full profit in real-time applications is only possible with adequate processors.

The book is divided into three main sections.

Part I (*A Unified Algebraic Approach for Classical Geometries*) introduces Euclidean, spherical and hyperbolic geometry in the frame of GA. Also the geometric modelling capabilities of GA from a general point of view are outlined. In this first part it will become clear that the language of GA is developing permanently and that by shaping this language, it can be adapted to the problems at hand.

David Hestenes, Hongbo Li and Alyn Rockwood summarize in chapter 1 the basic methods, ideas and rules of GA. This survey will be helpful for the reader as a general reference for all other chapters. Of chapters 2, 3, and 4, written by Hongbo Li et al., I especially want to emphasize two aspects. Firstly, the use of the so-called conformal split as introduced by Hestenes [110] in geometric modelling. Secondly, the proposed unification of classical geometries will become important in modelling catadioptic camera systems (see [GD00]), possessing both reflective and refractive components, for robot vision in a general framework. In chapter 6 Leo Dorst gives an introduction

[2] This research was funded since 1997 by the Deutsche Forschungsgemeinschaft.

to GA which will help to increase the influence of GA on many fields in computer science. Particularly interesting is his discussion of a very general filter scheme. Ambjörn Naeve and Lars Svensson present their own way of constructing GA in chapter 5. Working in the field of computer vision, they choose to demonstrate their framework in applications to geometrical optics.

Part II (*Algebraic Embedding of Signal Theory and Neural Computation*) is devoted to the development of a linear theory of intrinsically multidimensional signals and to make Clifford groups accessible in neural computations with the aim of developing neural networks as experts of basic geometric transformations and thus of the shape of objects.

This part is opened by a contribution of Valeri Labunets and his daughter Ekaterina Rundblad-Labunets, both representing the Russian School of algebraists. In chapter 7 they emphasize two important aspects of image processing in the CA framework. These are the modelling of vector-valued multidimensional signal data, including colour signals, and the formulation of invariants with that respect. Their framework is presented in a very general setting and, hopefully, will be picked up by other researchers to study its application.

The other six chapters of part II are written by the Kiel Cognitive Systems Group. In chapters 8 to 11 Thomas Bülow, Michael Felsberg and Gerald Sommer, partially in cooperation with Vladimir Chernov, Samara (Russia) for the first time are extensively presenting the way to represent intrinsically multidimensional scalar-valued signals in a linear manner by using a GA embedding. Several aspects are considered, as non-commutative and commutative hypercomplex Fourier transforms (chapters 8,9), fast algorithms for their computation (chapter 10), and local, hypercomplex signal representations in chapter 11. As a field of application of the proposed quaternion-valued Gabor transform in the two dimensional case, the problem of texture analysis is considered. In that chapter the old problems of signal theory as missing phase concepts of intrinsically two dimensional signals, embedded in 2D space, and the missing completeness of local symmetry representation (both problems have the same roots) could be overcome. Thus, the way to develop a linear signal theory of intrinsically multidimensional signals is prepared for future research.

Quite a different topic is handled by Sven Buchholz and Gerald Sommer in chapters 12 and 13. This is the design of neurons and neural nets (MLPs) which perform computations in CA. The new quality with respect to modelling neural computation results from the fact that the use of the geometric product in vector spaces induces a structural bias into the neurons. Looking onto the data through the glasses of "CA-neurons" gives valuable contraints while learning the intrinsic (geometric) structure of the data, which results in an excellent generalization ability. As a nearly equally important aspect the complexity of computations is drastically reduced because of the linearization effects of the algebraic embedding. These nets indeed constitute experts for geometric transformations.

Part III (*Geometric Algebra for Computer Vision and Robotics*) is concerned with actual topics of projective geometry in modelling computer vision tasks (chapters 14 to 17) and with the linear modelling of kinematic chains of points and lines in space (chapters 18 to 21).

In chapter 14, Christian Perwass and Joan Lasenby demonstrate a geometrically intuitive way of using the incidence algebra of projective geometry in GA to describe multiple view geometry. Especially the use of reciprocal frames should be emphasized. Many relations which have been derived in matrix algebra and Grassmann-Cayley algebra in the last years can be found here again. An application with respect to 3D reconstruction using vanishing points is laid out in chapter 15. Another application is demonstrated in chapter 16 by Eduardo Bayro-Corrochano and Bodo Rosenhahn with respect to the computation of the intrinsic parameters of a camera. Using the idea of the absolute conic in the context of Pascal's theorem, they develop a method which is comparable to the use of Kruppa equations.

Hongbo Li and Gerald Sommer present in chapter 17 an alternative way to chapter 14 of formulating multiple view geometry. In their approach they use a coordinate-free representation whereby image points are given as bivectors. Using this approach, they discovered new constraints on the trifocal tensor.

Chapters 18-21 are concerned with kinematics. In chapter 18, Eduardo Bayro-Corrochano is developing the framework of screw geometry in the language of motor algebra, a degenerate algebra isomorphic to that of dual quaternions. In contrast to dual quaternions, motors relate translation and rotation as spinors and, thus, result in some cases in simpler expressions. This is the case especially in considering kinematic chains, as is done by Eduardo Bayro-Corrochano and Detlev Kähler in chapter 19. They are modelling the forward and the inverse kinematics of robot arms in that framework. The use of dual quaternions with respect to motion alignment is studied as a tutorial paper by Kostas Daniilidis in chapter 20. His experience with this framework is based on a very successful application with respect to the hand-eye calibration problem in robot vision.

Finally, in chapter 21, Yiwen Zhang, Gerald Sommer and Eduardo Bayro-Corrochano are designing an extended Kalman filter for the tracking of lines. Because the motion of lines is intrinsical to the motor algebra, the authors can demonstrate the performance based on direct observations of such higher order entities. The presented approach can be considered as 3D-3D pose estimation based on lines. A more extensive study of 2D-3D pose estimation based on geometric constraints can be found in [SRZ00].

In summary, this book can serve as a reference of the actual state of applying Clifford algebra as a frame for geometric computing. Furthermore, it shows that the matter is alive and will hopefully grow and mature fast.

Thus, this book is also to be seen as a snapshot of current research and hence as a "workbench" for further developments in geometric computing.

To complete a project like this book requires the cooperation of the contributing authors. My thanks go to all of them. In particular I would like to thank Michael Felsberg for his substantial help with the coordination of this book project. He also prepared the final layout of this book with the help of the student Thomas Jäger. Many thanks to him, as well.

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