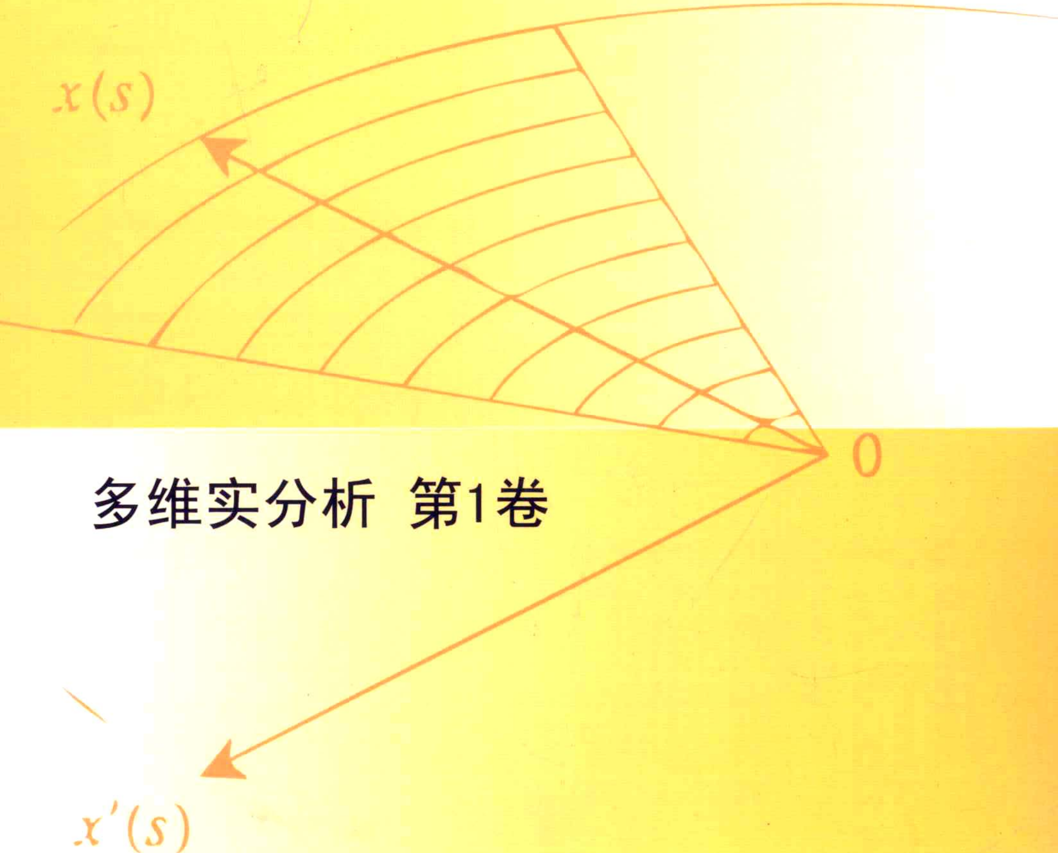
J. J. Duistermaat, J. A. C. Kolk

Multidimensional Real Analysis I

Differentiation



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MULTIDIMENSIONAL REAL ANALYSIS I:

DIFFERENTIATION

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To Saskia and Floortje

With Gratitude and Love

Preface

I prefer the open landscape under a clear sky with its depth of perspective, where the wealth of sharply defined nearby details gradually fades away towards the horizon.

This book, which is in two parts, provides an introduction to the theory of vector-valued functions on Euclidean space. We focus on four main objects of study and in addition consider the interactions between these. Volume I is devoted to differentiation. Differentiable functions on $\mathbf{R}^n$ come first, in Chapters 1 through 3. Next, differentiable manifolds embedded in $\mathbf{R}^n$ are discussed, in Chapters 4 and 5. In Volume II we take up integration. Chapter 6 deals with the theory of n-dimensional integration over $\mathbf{R}^n$. Finally, in Chapters 7 and 8 lower-dimensional integration over submanifolds of $\mathbf{R}^n$ is developed; particular attention is paid to vector analysis and the theory of differential forms, which are treated independently from each other. Generally speaking, the emphasis is on geometric aspects of analysis rather than on matters belonging to functional analysis.

In presenting the material we have been intentionally concrete, aiming at a thorough understanding of Euclidean space. Once this case is properly understood, it becomes easier to move on to abstract metric spaces or manifolds and to infinite-dimensional function spaces. If the general theory is introduced too soon, the reader might get confused about its relevance and lose motivation. Yet we have tried to organize the book as economically as we could, for instance by making use of linear algebra whenever possible and minimizing the number of ϵ–δ arguments, always without sacrificing rigor. In many cases, a fresh look at old problems, by ourselves and others, led to results or proofs in a form not found in current analysis textbooks. Quite often, similar techniques apply in different parts of mathematics; on the other hand, different techniques may be used to prove the same result. We offer ample illustration of these two principles, in the theory as well as the exercises.

A working knowledge of analysis in one real variable and linear algebra is a prerequisite. The main parts of the theory can be used as a text for an introductory course of one semester, as we have been doing for second-year students in Utrecht during the last decade. Sections at the end of many chapters usually contain applications that can be omitted in case of time constraints.

This volume contains 334 exercises, out of a total of 568, offering variations and applications of the main theory, as well as special cases and openings toward applications beyond the scope of this book. Next to routine exercises we tried also to include exercises that represent some mathematical idea. The exercises are independent from each other unless indicated otherwise, and therefore results are sometimes repeated. We have run student seminars based on a selection of the more challenging exercises.

In our experience, interest may be stimulated if from the beginning the student can perceive analysis as a subject intimately connected with many other parts of mathematics and physics: algebra, electromagnetism, geometry, including differential geometry, and topology, Lie groups, mechanics, number theory, partial differential equations, probability, special functions, to name the most important examples. In order to emphasize these relations, many exercises show the way in which results from the aforementioned fields fit in with the present theory; prior knowledge of these subjects is not assumed, however. We hope in this fashion to have created a landscape as preferred by Weyl,[1] thereby contributing to motivation, and facilitating the transition to more advanced treatments and topics.

[1] Weyl, H.: *The Classical Groups*. Princeton University Press, Princeton 1939, p. viii.

Acknowledgments

Since a text like this is deeply rooted in the literature, we have refrained from giving references. Yet we are deeply obliged to many mathematicians for publishing the results that we use freely. Many of our colleagues and friends have made important contributions: E. P. van den Ban, F. Beukers, R. H. Cushman, W. L. J. van der Kallen, H. Keers, M. van Leeuwen, E. J. N. Looijenga, D. Siersma, T. A. Springer, J. Stienstra, and in particular J. D. Stegeman and D. Zagier. We were also fortunate to have had the moral support of our special friend V. S. Varadarajan. Numerous small errors and stylistic points were picked up by students who attended our courses; we thank them all.

With regard to the manuscript's technical realization, the help of A. J. de Meijer and F. A. M. van de Wiel has been indispensable, with further contributions coming from K. Barendregt and J. Jaspers. We have to thank R. P. Buitelaar for assistance in preparing some of the illustrations. Without LaTeX, Y&Y TeX and Mathematica this work would never have taken on its present form.

J. P. van Braam Houckgeest translated the manuscript from Dutch into English. We are sincerely grateful to him for his painstaking attention to detail as well as his many suggestions for improvement.

We are indebted to S. J. van Strien to whose encouragement the English version is due; and furthermore to R. Astley and J. Walthoe, our editors, and to F. H. Nex, our copy-editor, for the pleasant collaboration; and to Cambridge University Press for making this work available to a larger audience.

Of course, errors still are bound to occur and we would be grateful to be told of them, at the e-mail address kolk@math.uu.nl. A listing of corrections will be made accessible through http://www.math.uu.nl/people/kolk.

Introduction

Motivation. Analysis came to life in the number space $\mathbf{R}^n$ of dimension n and its complex analog $\mathbf{C}^n$. Developments ever since have consistently shown that further progress and better understanding can be achieved by generalizing the notion of space, for instance to that of a manifold, of a topological vector space, or of a scheme, an algebraic or complex space having infinitesimal neighborhoods, each of these being defined over a field of characteristic which is 0 or positive. The search for unification by continuously reworking old results and blending these with new ones, which is so characteristic of mathematics, nowadays tends to be carried out more and more in these newer contexts, thus bypassing $\mathbf{R}^n$. As a result of this the uninitiated, for whom $\mathbf{R}^n$ is still a difficult object, runs the risk of learning analysis in several real variables in a suboptimal manner. Nevertheless, to quote F. and R. Nevanlinna: "The elimination of coordinates signifies a gain not only in a formal sense. It leads to a greater unity and simplicity in the theory of functions of arbitrarily many variables, the algebraic structure of analysis is clarified, and at the same time the geometric aspects of linear algebra become more prominent, which simplifies one's ability to comprehend the overall structures and promotes the formation of new ideas and methods".[2]

In this text we have tried to strike a balance between the concrete and the abstract: a treatment of differential calculus in the traditional $\mathbf{R}^n$ by efficient methods and using contemporary terminology, providing solid background and adequate preparation for reading more advanced works. The exercises are tightly coordinated with the theory, and most of them have been tried out during practice sessions or exams. Illustrative examples and exercises are offered in order to support and strengthen the reader's intuition.

Organization. In a subject like this with its many interrelations, the arrangement of the material is more or less determined by the proofs one prefers to or is able to give. Other ways of organizing are possible, but it is our experience that it is not such a simple matter to avoid confusing the reader. In particular, because the Change of Variables Theorem in Volume II is about diffeomorphisms, it is necessary to introduce these initially, in the present volume; a subsequent discussion of the Inverse Function Theorems then is a plausible inference. Next, applications in geometry, to the theory of differentiable manifolds, are natural. This geometry in its turn is indispensable for the description of the boundaries of the open sets that occur in Volume II, in the Theorem on Integration of a Total Derivative in $\mathbf{R}^n$, the generalization to $\mathbf{R}^n$ of the Fundamental Theorem of Integral Calculus on $\mathbf{R}$. This is why differentiation is treated in this first volume and integration in the second. Moreover, most known proofs of the Change of Variables Theorem require an Inverse Function, or the Implicit Function Theorem, as does our first proof. However, for the benefit of those readers who prefer a discussion of integration at

[2] Nevanlinna, F., Nevanlinna, R.: *Absolute Analysis.* Springer-Verlag, Berlin 1973, p. 1.

an early stage, we have included in Volume II a second proof of the Change of Variables Theorem by elementary means.

On some technical points. We have tried hard to reduce the number of ϵ–δ arguments, while maintaining a uniform and high level of rigor. In the theory of differentiability this has been achieved by using a reformulation of differentiability due to Hadamard.

The Implicit Function Theorem is derived as a consequence of the Local Inverse Function Theorem. By contrast, in the exercises it is treated as a result on the conservation of a zero for a family of functions depending on a finite-dimensional parameter upon variation of this parameter.

We introduce a submanifold as a set in $\mathbf{R}^n$ that can locally be written as the graph of a mapping, since this definition can easily be put to the test in concrete examples. When the "internal" structure of a submanifold is important, as is the case with integration over that submanifold, it is useful to have a description as an image under a mapping. If, however, one wants to study its "external" structure, for instance when it is asked how the submanifold lies in the ambient space $\mathbf{R}^n$, then a description in terms of an inverse image is the one to use. If both structures play a role simultaneously, for example in the description of a neighborhood of the boundary of an open set, one usually flattens the boundary locally by means of a variable $t \in \mathbf{R}$ which parametrizes a motion transversal to the boundary, that is, one considers the boundary locally as a hyperplane given by the condition $t = 0$.

A unifying theme is the similarity in behavior of global objects and their associated infinitesimal objects (that is, defined at the tangent level), where the latter can be investigated by way of linear algebra.

Exercises. Quite a few of the exercises are used to develop secondary but interesting themes omitted from the main course of lectures for reasons of time, but which often form the transition to more advanced theories. In many cases, exercises are strung together as projects which, step by easy step, lead the reader to important results. In order to set forth the interdependencies that inevitably arise, we begin an exercise by listing the other ones which (in total or in part only) are prerequisites as well as those exercises that use results from the one under discussion. The reader should not feel obliged to completely cover the preliminaries before setting out to work on subsequent exercises; quite often, only some terminology or minor results are required. In the review exercises we have primarily collected results from real analysis in one variable that are needed in later exercises and that might not be familiar to the reader.

Notational conventions. Our notation is fairly standard, yet we mention the following conventions. Although it will often be convenient to write column vectors as row vectors, the reader should remember that all vectors are in fact column vectors, unless specified otherwise. Mappings always have precisely defined domains and images, thus $f : \mathrm{dom}(f) \to \mathrm{im}(f)$, but if we are unable, or do not wish, to specify the domain we write $f : \mathbf{R}^n \supset\!\!\to \mathbf{R}^p$ for a mapping that is well-defined on some subset of $\mathbf{R}^n$ and takes values in $\mathbf{R}^p$. We write $\mathbf{N}_0$ for $\{0\} \cup \mathbf{N}$, $\mathbf{N}_\infty$ for $\mathbf{N} \cup \{\infty\}$, and $\mathbf{R}_+$ for $\{ x \in \mathbf{R} \mid x > 0 \}$. The open interval $\{ x \in \mathbf{R} \mid a < x < b \}$ in $\mathbf{R}$ is denoted by $]\,a, b\,[$ and not by (a, b), in order to avoid confusion with the element $(a, b) \in \mathbf{R}^2$.

Making the notation consistent and transparent is difficult; in particular, every way of designating partial derivatives has its flaws. Whenever possible, we write $D_j f$ for the j-th column in a matrix representation of the total derivative Df of a mapping $f : \mathbf{R}^n \to \mathbf{R}^p$. This leads to expressions like $D_j f_i$ instead of Jacobi's classical $\frac{\partial f_i}{\partial x_j}$, etc. As a bonus the notation becomes independent of the designation of the coordinates in $\mathbf{R}^n$, thus avoiding absurd formulae such as may arise on substitution of variables; a disadvantage is that the formulae for matrix multiplication look less natural. The latter could be avoided with the notation $D_j f^i$, but this we rejected as being too extreme. The convention just mentioned has not been applied dogmatically; in the case of special coordinate systems like spherical coordinates, Jacobi's notation is the one of preference. As a further complication, D_j is used by many authors, especially in Fourier theory, for the momentum operator $\frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$.

We use the following dictionary of symbols to indicate the ends of various items:

❑ Proof
❍ Definition
✰ Example

Contents

Volume II

Chapter 1

Continuity

Continuity of mappings between Euclidean spaces is the central topic in this chapter. We begin by discussing those properties of the n-dimensional space $\mathbf{R}^n$ that are determined by the standard inner product. In particular, we introduce the notions of distance between the points of $\mathbf{R}^n$ and of an open set in $\mathbf{R}^n$; these, in turn, are used to characterize limits and continuity of mappings between Euclidean spaces. The more profound properties of continuous mappings rest on the completeness of $\mathbf{R}^n$, which is studied next. Compact sets are infinite sets that in a restricted sense behave like finite sets, and their interplay with continuous mappings leads to many fundamental results in analysis, such as the attainment of extrema as well as the uniform continuity of continuous mappings on compact sets. Finally, we consider connected sets, which are related to intermediate value properties of continuous functions.

In applications of analysis in mathematics or in other sciences it is necessary to consider mappings depending on more than one variable. For instance, in order to describe the distribution of temperature and humidity in physical space–time we need to specify (in first approximation) the values of both the temperature T and the humidity h at every $(x, t) \in \mathbf{R}^3 \times \mathbf{R} \simeq \mathbf{R}^4$, where $x \in \mathbf{R}^3$ stands for a position in space and $t \in \mathbf{R}$ for a moment in time. Thus arises, in a natural fashion, a mapping $f : \mathbf{R}^4 \to \mathbf{R}^2$ with $f(x, t) = (T, h)$. The first step in a closer investigation of the properties of such mappings requires a study of the space $\mathbf{R}^n$ itself.

1.1 Inner product and norm

Let $n \in \mathbf{N}$. The n-dimensional space $\mathbf{R}^n$ is the Cartesian product of n copies of the linear space $\mathbf{R}$. Therefore $\mathbf{R}^n$ is a linear space; and following the standard convention in linear algebra we shall denote an element $x \in \mathbf{R}^n$ as a column vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R}^n.$$

For typographical reasons, however, we often write $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, or, if necessary, $x = (x_1, \dots, x_n)^t$ where ^{t} denotes the transpose of the $1 \times n$ matrix. Then $x_j \in \mathbf{R}$ is the j-th *coordinate* or *component* of x.

We recall that the *addition* of vectors and the *multiplication* of a vector by a *scalar* are defined by components, thus for $x, y \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$

$$\begin{aligned} (x_1, \dots, x_n) + (y_1, \dots, y_n) &= (x_1 + y_1, \dots, x_n + y_n), \\ \lambda(x_1, \dots, x_n) &= (\lambda x_1, \dots, \lambda x_n). \end{aligned}$$

We say that $\mathbf{R}^n$ is a *vector space* or a *linear space* over $\mathbf{R}$ if it is provided with this addition and scalar multiplication. This means the following. Vector addition satisfies the commutative group axioms: *associativity* ($(x + y) + z = x + (y + z)$), *existence of zero* ($x + 0 = x$), *existence of additive inverses* ($x + (-x) = 0$), *commutativity* ($x + y = y + x$); scalar multiplication is *associative* ($(\lambda\mu)x = \lambda(\mu x)$) and *distributive* over addition in both ways, i.e. ($\lambda(x + y) = \lambda x + \lambda y$ and $(\lambda + \mu)x = \lambda x + \mu x$). We assume the reader to be familiar with the basic theory of finite-dimensional linear spaces.

For mappings $f : \mathbf{R}^n \supset\!\!\rightarrow \mathbf{R}^p$, we have the *component functions* $f_i : \mathbf{R}^n \supset\!\!\rightarrow \mathbf{R}$, for $1 \le i \le p$, satisfying

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} : \mathbf{R}^n \supset\!\!\rightarrow \mathbf{R}^p.$$

Many geometric concepts require an extra structure on $\mathbf{R}^n$ that we now define.

Definition 1.1.1. The *Euclidean space* $\mathbf{R}^n$ is the aforementioned linear space $\mathbf{R}^n$ provided with the *standard inner product*

$$\langle x, y \rangle = \sum_{1 \le j \le n} x_j y_j \qquad (x, y \in \mathbf{R}^n).$$

In particular, we say that x and $y \in \mathbf{R}^n$ are mutually *orthogonal* or *perpendicular* vectors if $\langle x, y \rangle = 0$. ❍

The standard inner product on $\mathbf{R}^n$ will be used for introducing the notion of a distance on $\mathbf{R}^n$, which in turn is indispensable for the definition of limits of mappings defined on $\mathbf{R}^n$ that take values in $\mathbf{R}^p$. We list the basic properties of the standard inner product.

Lemma 1.1.2. *All $x, y, z \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ satisfy the following relations.*

(i) Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.

(ii) Linearity: $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$.

(iii) Positivity: $\langle x, x \rangle \geq 0$, *with equality if and only if* $x = 0$.

Definition 1.1.3. The *standard basis* for $\mathbf{R}^n$ consists of the vectors

$$e_j = (\delta_{1j}, \dots, \delta_{nj}) \in \mathbf{R}^n \qquad (1 \leq j \leq n),$$

where δ_{ij} equals 1 if $i = j$ and equals 0 if $i \neq j$. ❍

Thus we can write

$$x = \sum_{1 \leq j \leq n} x_j e_j \qquad (x \in \mathbf{R}^n). \tag{1.1}$$

With respect to the standard inner product on $\mathbf{R}^n$ the standard basis is *orthonormal*, that is, $\langle e_i, e_j \rangle = \delta_{ij}$, for all $1 \leq i, j \leq n$. Thus, $\|e_j\| = 1$, while e_i and e_j, for distinct i and j, are mutually orthogonal vectors.

Definition 1.1.4. The *Euclidean norm* or *length* $\|x\|$ of $x \in \mathbf{R}^n$ is defined as

$$\|x\| = \sqrt{\langle x, x \rangle}.$$ ❍

From this definition and Lemma 1.1.2 we directly obtain

Lemma 1.1.5. *All $x, y \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ satisfy the following properties.*

(i) Positivity: $\|x\| \geq 0$, *with equality if and only if* $x = 0$.

(ii) Homogeneity: $\|\lambda x\| = |\lambda| \, \|x\|$.

(iii) Polarization identity: $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$, *which expresses the standard inner product in terms of the norm.*

(iv) Pythagorean property: $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$ *if and only if* $\langle x, y \rangle = 0$.

Proof. For (iii) and (iv) note

$$\begin{aligned} \|x \pm y\|^2 &= \langle x \pm y, x \pm y \rangle = \langle x, x \rangle \pm \langle x, y \rangle \pm \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2 \pm 2\langle x, y \rangle. \end{aligned}$$ ❑