FOUNDATIONS OF GEOMETRY

C. R. WYLIE, JR.

FOUNDATIONS OF GEOMETRY

C. R. WYLIE, JR.
Professor and Chairman
Department of Mathematics
University of Utah

FOUNDATIONS OF GEOMETRY

Copyright © 1964 by McGraw-Hill, Inc. All Rights Reserved. Printed in the United States of America. This book, or parts thereof, may not be reproduced in any form without permission of the publishers.

Library of Congress Catalog Card Number 63-23049

42183

PREFACE

This book has been written primarily for students preparing to become teachers of secondary school mathematics, although it should also be of interest to practicing teachers and to undergraduates majoring in mathematics who wish to review or to extend their background in geometry. Its purpose is to present a careful axiomatic development of certain important parts of elementary euclidean and non-euclidean geometry, and, in so doing, to acquaint the student with the axiomatic method as a general pattern of thought. To some, the word "Foundations" in the title of a book suggests a boring preoccupation with details or an intensive belaboring of the obvious. However, to the prospective reader of this book, I would express both the conviction that its concern with details, while extensive, is not significantly greater than that found in the new high school programs, and the hope that such concern is adequately motivated and properly balanced by exciting glimpses of what lies beyond the bounds of traditional euclidean geometry.

The book begins with a chapter on the axiomatic method and its major features, independent of its use in geometry. Then, in Chapter 2, postulates for three-dimensional euclidean geometry are introduced and the principal results up to, but not including, the measurement of volume are carefully developed. Chapter 3, though it contains nothing original, is perhaps the most novel in the book. It is devoted to an axiomatic development of the simpler aspects of four-dimensional euclidean geometry, and is intended to give the student practice in the axiomatic method in a setting, essentially as simple as the geometry of three dimensions, but in which he has no familiarity with the main results to guide or to hinder him. Chapter 4 provides an introduction to plane hyperbolic geometry and carries the development as far as the measurement of area. Finally, in Chapter 5 the question of the consistency of hyperbolic geometry is considered; and by describing in detail a model in the euclidean plane in which each of the postulates of hyperbolic geometry can be verified, it is shown that hyperbolic geometry is relatively consistent.

Those who are familiar with the geometry texts prepared by the School Mathematics Study Group will find a striking resemblance between the postulates employed in those books and the postulates adopted here. It was my great good fortune to participate in the writing of the School Mathematics Study Group's "Geometry with Coordinates," and the present book owes much to the stimulating contacts I had with my colleagues in this project. Of course, the School Mathematics Study Group has no responsibility for what I have written here, and my obvious indebtedness to it does not imply any indorsement of my work. However, it is a pleasant obligation on my part to acknowledge the kindness of the Group and its Director, Professor E. G. Begle, in permitting me to employ its wording of a number of postulates and theorems and to use, without change, occasional passages which I wrote during the preparation of "Geometry with Coordinates."

The author of any textbook owes much to his own teachers, colleagues, and students; and to all who have assisted me, consciously or unconsciously, in the preparation of this book, I express my appreciation. Finally, it is a pleasure to acknowledge the welcome assistance of Miss Maxine Winterton, who typed the manuscript, and of my wife, Ellen, and my secretary, Mrs. Patricia Everts, who shared the task of reading the proof.

C. R. Wylie, Jr.

试读结束：需要全本请在线购买： www.ertongbook.com

CONTENTS

Chapter 3
The Geometry of Four Dimensions 163

Chapter 4
Plane Hyperbolic Geometry 207

Chapter 5
A Euclidean Model of the Hyperbolic Plane 290

1 THE AXIOMATIC METHOD

1.1 Introduction. The origin of the discipline we call geometry is clearly evident in the name itself, which derives from the Greek words *ge*, meaning *the earth*, and *metrein*, meaning *to measure*. In the beginning geometry was, indeed, the art (not science) of earth measurement and consisted of a disorganized collection of rules for computing simple areas and volumes and carrying out a few elementary constructions. Such results were the fruits of long centuries of trial and error by the Babylonians and the Egyptians who, in the dawn of civilization, had to develop practical procedures for such things as land surveying, the construction of granaries and canals, and the erection of tombs and temples. Some of this information was correct and some incorrect though useful as an approximation, but all had at best only the sanction of plausibility. In other words, for the first two thousand years or so of its existence, geometry was a body of empirical knowledge obtained inductively from a consideration of many special cases and completely unsupported by anything resembling logical proof.

Then in the millennium immediately preceding the Christian era, geometry underwent a remarkable change. The Greeks, inclined by temperament toward philosophy and abstraction and blessed with security and leisure to follow these inclinations, took the geometry of the Egyptians and recast it in the form of a deductive science. Beginning with Thales (Thā′-lēz, 640–546 B.C.), this transformation culminated in the work of Euclid (365?–275? B.C.), whose "Elements" presented the sum total of current geometrical knowledge, not as a disjointed collection of empirical results, but as a well-organized chain of theorems following inevitably by the laws of logic from a few simple initial assumptions. Euclid's "Elements" remains the most famous and most important textbook ever written. Not only did it permanently establish the character of geometry as a deductive science, it also exemplified a pattern of logical organization so effective and so elegant that today most of mathematics is constructed according to the same plan. In major areas of other disciplines, such as biology, chemistry, economics, physics, and psychology, the goal of scholars is still to achieve a comparable logical structure.

The abstract logical plan which Euclid conceived and so ably illustrated in his "Elements" we now refer to as the **axiomatic method,** and any particular instance of it we call an **axiomatic system.** The impact of the axiomatic method upon mathematics, and upon other sciences as well, has been so profound that not only the scholar who must be prepared to use it in his own work but also the intelligent layman who would achieve some understanding of the nature of scientific thought must be familiar with it. Accordingly, we shall devote the balance of this chapter to a discussion of the axiomatic method and its major features.

1.2 Inductive and Deductive Reasoning. A careless reading of the preceding section might well leave one with the impression that inductive reasoning is primitive and unscientific and that deductive reasoning is the only mode of thought appropriate to genuine scientific inquiry. Nothing could be further from the truth. Although this book is devoted almost exclusively to instances of deductive reasoning, we should understand from the outset the nature of induction and the important role which it plays in every science, including mathematics.

Induction can be described briefly as the process of inferring general properties, relations, or laws from particular instances which have been observed. Deduction, on the other hand, is the process of reasoning to particular conclusions from general principles that have been accepted as the starting point of an argument. Both induction and deduction have their merits and their defects, and neither by itself is sufficient to support genuine scientific progress.

Induction has at least two obvious weaknesses. In the first place, no matter how many completely correct observations have been made, unless every possible instance has been examined, no generalization can be made with certainty, because any of the uninvestigated cases may contradict it. Second, the assumption that the observations actually made have been made with perfect accuracy is often false, so that in many cases there is not exact information on which to base a generalization.

To illustrate, if we evaluate the expression $n^2 - n + 17$ for the first few positive integers we obtain the following table:

n	$n^2 - n + 17$
1	17
2	19
3	23
4	29
5	37
6	47
7	59
8	73
9	89
10	107

Here, we are in the fortunate position of having a set of completely accurate observations from which to generalize, and by induction we may be led to any of several plausible conjectures. For instance, from the particular cases before us, we may draw the almost obvious conclusion that for every positive integer n the expression $n^2 - n + 17$ is a number which is odd. Or, taking a somewhat closer look, we may conclude further that for every positive integer n the expression $n^2 - n + 17$ is a number which ends in 3, 7, or 9. Or, observing that 17, 19, 23, . . . , 107 are all prime numbers, we may infer the still more remarkable property that $n^2 - n + 17$ is a prime number for every positive integer n. Each of these conclusions is strongly suggested by the data. But are they all correct, and if so, how can we be sure? Trying additional values of n *may* answer the question, for if we find for some n that $n^2 - n + 17$ is not odd, or does not end in one of the digits 3, 7, or 9, or is not a prime number, then the corresponding inference is immediately overthrown by that one counterexample. But if we investigate additional cases and find them all consistent with our conjectures, the issue remains in doubt. We may feel that the additional supporting examples increase the probability that our generalizations are correct, but we still must admit the possibility that among the cases not yet examined there may be at least one which will contradict, and hence overthrow, one or more of our conjectures. Specifically, if we extend our table a bit, we find

n	$n^2 - n + 17$
11	127
12	149
13	173
14	199
15	227
16	257
17	289

Through $n = 16$, the entries support each of our conjectures. However, for $n = 17$ we find that $n^2 - n + 17 = 289$ is not a prime but in fact is equal to $(17)^2$. Thus our third conjecture is false, while the other two, though perhaps more plausible because of the additional supporting evidence, remain uncertain.

It is precisely at this point that deduction "comes to the rescue" and "takes over" from induction. When further search for a counterexample seems fruitless and the evidence of a sufficient number of supporting examples convinces a mathematician that a conjecture is probably true, he abandons induction and tries to *prove* the conjecture by deriving or deducing it from more fundamental principles. In particular, though we shall not digress to do so, it is easy to deduce from the principles of arithmetic that for every positive integer n the expression $n^2 - n + 17$ is a number which is odd and which moreover ends in 3, 7, or 9.

On the other hand, deduction has its weaknesses. In the first place, deduction can only provide us with conditional statements of the form "*If* something is true, *then* something else is true." It is essentially unconcerned with whether or not the statements with which an argument begins are true or false. Second, deduction is, in itself, incapable of providing us with either the results which we hope to prove or the initial statements from which we propose to evolve a proof.

It is here that induction steps in and saves deduction from its inherent sterility. It is induction that suggests the theorems which deduction subsequently so painstakingly tries to prove. It is induction, too, which provides us with the insights that we formalize in the principles on which our proofs are based. And insofar as these principles have any claim to truth, it is induction which ultimately supports that claim. Without the body of geometric information accumulated by the Babylonians and the Egyptians and his Greek predecessors, Euclid would have had no material to organize, no results to set in logical order, no initial principles from which to reason. The proofs of theorems are achievements of deductive reasoning, but theorems themselves are the fruits of induction, of intuition, of creative insight habitually examining every special case for suggestions of more general relations.

On its lower levels, induction is merely the tedious cataloging of observations; at its best, it is the imaginative recognition, through a thousand irrelevancies, of the essential nature of a situation. Without induction, deduction can only wait, idly, for something to prove. Without deduction, induction is always unsure of itself, its inferences suspect, its insights, no matter how brilliant, vulnerable to counterexamples and disproof. And in more practical terms, without skill in both induction and deduction, that is, without both intuition and a

试读结束：需要全本请在线购买： www.ertongbook.com

clear feeling for proof, no student of mathematics is more than marginally prepared for his work.

The cooperative relation between induction and deduction is roughly comparable to the relation between mathematics and the other sciences. Historically, mathematics developed out of man's concern with physical problems. And though mathematics is ultimately a construction of the mind alone and exists only as a magnificent collection of ideas, from earliest times down to the present day it has been stimulated and inspired and enriched by contact with the external world. Its problems are often idealizations of problems first encountered by the physicist or the engineer, or, more recently, by the social scientist. Many of its concepts are abstractions from the common experience of all men. And many areas of mathematics stem originally from the needs of scholars for more powerful analytical tools with which to pursue their investigations of the world around them. Surely, without contact with the external world, mathematics, if it existed at all, would be vastly different from what it is today.

But mathematics repays generously her indebtedness to the other sciences. At the mere suggestion of a new problem, mathematics sets to work developing procedures for its solution, generalizing it, relating it to work already done and to results already known, until finally it gives back to scholars in the original field a well-developed theory for their use.

Oftentimes mathematics outruns completely the demands of the physical problem which may have stimulated it, and careless critics scoff at its "pure" or "abstract" or "useless" character. Such criticism is absurd on two quite different counts. In the first place, all mathematics worthy of the name is pure or abstract or even, in a certain sense, useless, just as poetry and music and painting and sculpture are useless except as they bring satisfaction to those who create and to those who enjoy. It is no more appropriate to criticize mathematics for possessing the attributes of one of the creative arts than it is to criticize the arts themselves, unless it be that perhaps the number of those who find enjoyment in mathematics, though considerable, is less than those who enjoy the more conventional arts. Then in the second place, by a remarkable coincidence which is almost completely responsible for the existence of all present-day science and technology and which has no counterpart in the arts, even the most abstract parts of mathematics have turned out again and again to be highly "practical," as science, becoming ever more sophisticated, finds that it needs mathematical tools of greater and greater refinement.

As we now begin to employ the deductive method in our investigation of one small part of the great field of mathematics, it is important that we do not lose sight of the great significance of the inductive method, which suggested most, if not all, of the results with which we shall be concerned. And it is important, too, that even while we remain thoroughly committed to the conviction that geometry is a branch of pure mathematics, we never forget that it had its origins in the physical world, owes much of its vitality to its contacts with the physical world, and, properly understood, is a tool of magnificent power and effectiveness for the study of the physical world.

EXERCISES

1. Prove that for all integral values of n, $n^2 - n + 17$ is odd and ends in one of the digits 3, 7, or 9.
2. What do you think is the most plausible value for the next number in each of the following sequences?

   (*a*) 0, 0, 0, 0, . . .   (*b*) 1, 0, 1, 0, . . .
   (*c*) 1, 2, 3, 4, . . .   (*d*) 1, 2, 4, 8, . . .

   Can you prove your conjectures? Can you construct expressions which will yield the given values for $n = 1, 2, 3$, and 4 but in each case will yield the value π when $n = 5$?
3. Evaluate $x^7 - 14x^5 + 49x^3 - 39x$ for $x = 0, \pm 1, \pm 2, \ldots$ What generalizations occur to you? Are your inferences correct for all values of x? Are they correct for all integral values of x?
4. Determine the sum of the first k odd positive integers for a number of values of k. What generalizations occur to you? Are your inferences correct for all positive integers k?
5. Determine the sum of the cubes of the first k positive integers for a number of values of k. What generalizations occur to you? Are your inferences correct for all positive integers k?
6. Evaluate $n^2 - 39n + 421$ for a number of integral values of n. What generalizations occur to you? Are your inferences correct for all integral values of n?
7. Let a sequence of integers $u_1, u_2, u_3, \ldots, u_r, \ldots$ be defined by the conditions

$$u_1 = 1$$
$$u_2 = 1$$
$$u_{n+2} = u_{n+1} + 2u_n \qquad n = 1, 2, 3, \ldots$$

   What general properties of this sequence occur to you? Let Π_r denote the product of the first r numbers in the sequence and evaluate the expression

$$\frac{\Pi_r}{\Pi_k \Pi_{r-k}} \qquad 0 < k < r$$

   for various values of r and k. What generalizations occur to you?
8. Perform the following geometric "experiment" several times: Construct a quadrilateral of any shape and determine the midpoints of its sides. What properties of the four midpoints occur to you?

9. Perform the following geometric "experiment" several times: Construct two triangles so related that the lines joining corresponding vertices pass through the same point, and locate the point of intersection of the lines determined by corresponding sides of the two triangles. What properties of the three points of intersection occur to you?

10. Perform the following geometric "experiment" several times: Let l_1 and l_2 be two intersecting lines, and on each choose three points, say A_1, B_1, C_1 and A_2, B_2, C_2, distinct from the intersection of l_1 and l_2. Determine the intersection of each of the following pairs of lines:

$$A_1B_2 \text{ and } A_2B_1. \qquad A_2B_3 \text{ and } A_3B_2, \qquad A_1B_3 \text{ and } A_3B_1$$

What properties of the three points of intersection occur to you?

11. Repeat the "experiment" described in Exercise 10, only this time let the points A_1, B_1, C_1 and A_2, B_2, C_2 be chosen so that the lines A_1A_2, B_1B_2, C_1C_2 all pass through the same point. What properties of the three points of intersection defined in Exercise 10 occur to you in this case?

12. Perform the following geometric "experiment" several times: On any circle choose any six points, P_1, P_2, P_3, P_4, P_5, P_6, and determine the intersection of each of the following pairs of lines:

$$P_1P_2 \text{ and } P_4P_5, \qquad P_2P_3 \text{ and } P_5P_6, \qquad P_3P_4 \text{ and } P_6P_1$$

What properties of the three points of intersection occur to you? Can you think of a generalization of this "experiment" to curves other than circles?

13. Perform the following geometric "experiment" several times: Draw three circles, C_1, C_2, and C_3, so related that each one intersects each of the others at two distinct points. Draw the lines determined by the points of intersection of each pair of circles. What properties of the three lines occur to you? Can you think of a generalization of this "experiment" to curves other than circles?

14. Perform the following geometric "experiment" several times: Construct a triangle of any shape and on each of its sides construct an equilateral triangle having that side as base. For each of the three equilateral triangles, determine the point of intersection of its medians. What properties of these three points occur to you? Can you think of a generalization of this "experiment" to polygons other than triangles?

15. Discuss the following "proof" that 1 is the largest positive integer:

"If n is any positive integer except 1, it is obvious that n^2 is an integer which is still larger. Hence no positive integer different from 1 can be the largest and so, perforce, 1 must be the largest positive integer." What theorem, if any, is established by this argument?

16. Discuss the "moral," if any, to the following anecdote:

An engineer, a physicist, and a mathematician were once riding together through the sheep country of Montana. Glancing across the plains, the engineer saw a small flock of sheep and remarked, "Well, I see there are some black sheep in Montana." The physicist, looking out and observing that there was but a single black sheep in the little flock, rebuked the engineer, saying, "As scientists, don't you think we should say simply that there is at least one black sheep in Montana?" Then the mathematician, having made his survey of the flock, said, "Gentlemen, it appears to me that all we are entitled to assert is that there is at least one sheep in Montana which is black on one side."

17. Discuss the following statement of the principle of induction:

"If certain members of a class are observed to have a given property and if the rest of the members of the class have this property, then all members of the class have the given property."

Discuss each of the following quotations:

18. "We see that experience plays an indispensable role in the genesis of geometry; but it would be an error then to conclude that geometry is even in part an experimental science."

Henri Poincaré, "Science and Hypothesis," p. 79 [1].*

19. "As we emphasize the deductive structure of our science [mathematics] and of acceptable proof, let us not lose sight of the fact that many of the most significant results that we prove were arrived at by guess-work, by intuition, by brilliant insight."

Mina Rees, "The Nature of Mathematics," *Science*, Oct. 5, 1962, p. 11 [2].

20. "I address myself to all interested students of mathematics of all grades and I say: Certainly let us learn proving, but also *let us learn guessing*."

George Polya, "Induction and Analogy in Mathematics," p. v [3].

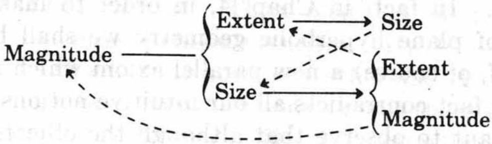
1.3 Axiomatic Systems. Before we can begin to organize our knowledge of geometry into a logical, deductive structure, we must first become familiar with the general features of the kind of organization we hope to achieve. Briefly, an axiomatic system consists of the following:

1. A set of undefined terms which forms the basis of the necessary technical vocabulary
2. A set of unproved initial assumptions
3. The laws of logic
4. The body of theorems, expressing properties of the undefined objects, which are derived from the axioms by the laws of logic

At first glance it might appear that in any logical discussion every term should be carefully defined, but a moment's reflection shows that this is impossible! New terms can be defined only by means of others already defined and understood. Thus, attempting to define every term either leads us to some first word, for whose definition no other words are available, or else leads us in circles in which, in effect, we define A in terms of B, B in terms of C, and C in terms of A! How

* Bracketed numbers refer to full bibliographic credits listed at the end of the chapter.

common the latter process is can be seen by looking up almost any word in a dictionary, then looking up the words used in its definition, and so on. Usually in just a few steps one reaches a definition in which the original word reappears! For instance, in one of the standard unabridged dictionaries we find the following chain of definitions purporting to give meaning to the term *magnitude:*



In the development of an axiomatic system the impossibility of defining every term is explicitly recognized and certain technical terms are deliberately left undefined. The vocabulary of the discussion then consists of these undefined terms, other technical terms defined by means of them, and the nontechnical vocabulary of everyday discourse which, of course, we implicitly assume to be available.

Similarly, it might be thought that in any scientific discussion every assertion should be carefully proved, but this, too, is impossible. If we attempt it, we either embark upon an infinite regression, in which we assert that

A is true because B is true.
B is true because C is true.
C is true because D is true.
. . . .

or else we reason in a circle and assert, in effect, that

A is true because B is true.
B is true because C is true.
C is true because A is true.

The impossibility of proving every statement is also explicitly recognized in the construction of an axiomatic system, and certain statements, nowadays referred to interchangeably as **axioms** or **postulates**, are accepted without proof as a necessary starting point for the discussion.

Occasionally it is said that axioms are facts which are taken for granted because their truth is so obvious that it needs no proof. Since the purpose of an axiomatic system is to provide an orderly development in which complicated and difficult results are deduced from simpler and more fundamental ones, the initial assumptions are often so simple that they do, indeed, seem obviously true, and on psychological and pedagogical grounds this is probably desirable. Moreover, in choosing among various sets of axioms which may serve equally well as the starting point of a deductive development, it is eminently proper to choose the one which seems most natural or most plausible. Nevertheless, the ultimate reason for accepting a set of axioms without proof is simply that no other course is possible and has nothing to do with the obviousness or intuitive appeal of the individual axioms. In fact, in Chap. 4, in order to make possible our development of plane hyperbolic geometry we shall have to accept (without proof, of course) a new parallel axiom which not only is not obvious but in fact contradicts all our intuitive notions of parallelism.

It is important to observe that although the objects and relations with which an axiomatic system deals are ultimately undefined, they are certainly not meaningless. In fact, the axioms make statements about them, and by means of the accepted laws of logic additional results, or theorems, are proved to be true about them. Thus, by the processes of deductive reasoning, more and more properties of the objects of the system become established, not just with experimental accuracy, but with the certainty that they follow as logical consequences of the initial assumptions.

It is sometimes said that a mathematician working with an axiomatic system is merely playing a meaningless game with undefined pieces subject to arbitrary rules, and in a very real sense this is true. However, it is not the whole truth, and without further qualification it is only a caricature of the truth. For as we pointed out in the last section, the construction of an axiomatic system is usually motivated by ideas drawn from the external world. The undefined terms are often idealizations suggested by objects in the world of our physical experience, and the axioms are abstract formulations of the fundamental observed properties of these objects. To abandon contact with the "real" world in this fashion may seem foolish to those of a practical turn of mind, but it is highly practical. It allows the instruments of the mind to replace the instruments of the hand and eye in the study of the phenomena of original interest. And if the initial abstraction from the world of experience was made with appropriate care, the results deduced from the axioms by the laws of logic can be transported back into the "real" world either as properties supported now by deduction as well as induction or often as new properties that were previously unknown. Moreover, the fact that an axiomatic system deals with undefined things means, in effect, that as we work with it we are "killing many birds with one stone." For an abstract system, though it may have been motivated or suggested by a specific set of objects or facts, is bound to no particular interpretation. Its results can be applied equally well to any system whose elements can be identi-

试读结束：需要全本请在线购买： www.ertongbook.com