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THE $\bar{\partial}$-NEUMANN PROBLEM AND SCHRÖDINGER OPERATORS

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Volume 59

To Hedi
Enar and Philipp
Kathi and Christopher

Preface

The subject of this book is complex analysis in several variables and its connections to partial differential equations and to functional analysis. The first sections of each chapter contain prerequisites from functional analysis, Sobolev spaces, partial differential equations and spectral analysis, which are used in the following sections devoted to the main topic of the book. In this way the book becomes self-contained, with only one exception, where we do not provide all details in the proof of the general spectral theorem for unbounded self-adjoint operators.

We concentrate on the Cauchy–Riemann equation ($\overline{\partial}$-equation) and investigate the properties of the canonical solution operator to $\overline{\partial}$, the solution with minimal L^2-norm and its relationship to the $\overline{\partial}$-Neumann operator. The first chapter contains a discussion of Bergman spaces in one and several complex variables, including basic facts on Hilbert spaces. In the second chapter the solution operator to $\overline{\partial}$ restricted to holomorphic L^2-functions in one complex variable is investigated, pointing out that the Bergman kernel of the associated Hilbert space of holomorphic functions plays an important role. We investigate operator properties like compactness and Schatten class membership, also for the solution operator on weighted spaces of entire functions (Fock spaces). In the third chapter we generalize the results to several complex variables and explain some new phenomena which do not appear in one variable.

In the following we consider the general $\overline{\partial}$-complex and derive properties of the complex Laplacian on L^2-spaces of bounded pseudoconvex domains and on weighted L^2-spaces. For this purpose we first concentrate on basic results about distributions, Sobolev spaces, and unbounded operators on Hilbert spaces. The key result in J. J. Kohn's far-reaching method is the Kohn–Morrey formula, which is presented in different versions. Using this formula the basic properties of the $\overline{\partial}$-Neumann operator – the bounded inverse of the complex Laplacian – are proved. In recent years it has turned out to be useful to investigate an even more general situation, namely the twisted $\overline{\partial}$-complex, where $\overline{\partial}$ is composed with a positive twist factor. In this way one obtains a rather general basic estimate, from which one gets Hörmander's L^2-estimates for the solution of the Cauchy–Riemann equation together with results on related weighted spaces of entire functions, such as that these spaces are infinite-dimensional if the eigenvalues of the Levi matrix of the weight function show a certain behavior at infinity. In addition, it is pointed out that some L^2-estimates for $\overline{\partial}$ can be interpreted in the sense of a general Brascamp–Lieb inequality.

The next chapter contains a detailed account of the application of the $\overline{\partial}$-methods to Schrödinger operators, Pauli and Dirac operators and to Witten–Laplacians. In this context, spectral analysis plays an important role. Therefore an extensive chapter on spectral analysis was inserted to provide a better understanding for the operator theoretic aspects in the $\overline{\partial}$-Neumann problem, which, in particular, is used to exactly describe the spectrum of complex Laplacian on the Fock space. Returning to the

$\bar{\partial}$-Neumann problem, we characterize compactness of the $\bar{\partial}$-Neumann operator using a description of precompact subsets in L^2-spaces. Compactness of the $\bar{\partial}$-Neumann operator is also related to properties of commutators of the Bergman projection and multiplication operators.

In the last part we use the $\bar{\partial}$-methods and some spectral theory to settle the question whether certain Schrödinger operators with a magnetic field have compact resolvent. It is also shown that a large class of Dirac operators fail to have compact resolvent. Finally we exhibit some situations where the $\bar{\partial}$-Neumann operator is not compact.

Numerous references for the topics of the text and for additional results are given in the notes at the end of each chapter.

Most of the material of the book stems from various lectures of the author given at the University of Vienna, the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna and at CIRM, Luminy, during programs on the $\bar{\partial}$-Neumann operator in recent years. The author is indebted to both institutions, ESI and CIRM, for their help and hospitality. I would also like to thank my students Franz Berger, Damir Ferizović and Tobias Preinerstorfer for their constructive criticisms of the manuscript, and also for their help in eliminating a number of typos and minor errors.

Vienna, March 2014 *Friedrich Haslinger*

Contents

1 Bergman spaces

To investigate the solution to the inhomogeneous $\bar{\partial}$-equation $\bar{\partial}u = g$, we will first consider the case where the right-hand side g is a holomorphic function. Therefore we need an appropriate Hilbert space of holomorphic functions – the Bergman space. We will use standard basic facts about Hilbert spaces, such as the Riesz representation theorem for continuous linear functionals, facts about orthogonal projections, and complete orthonormal bases.

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and the Bergman space

$$A^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \text{ holomorphic } : \|f\|^2 = \int_\Omega |f(z)|^2 \, d\lambda(z) < \infty\},$$

where λ is the Lebesgue measure of $\mathbb{C}^n$. The inner product is given by

$$(f, g) = \int_\Omega f(z)\,\overline{g(z)}\, d\lambda(z),$$

for $f, g \in A^2(\Omega)$.

1.1 Elementary properties

For sake of simplicity we first restrict ourselves to domains $\Omega \subseteq \mathbb{C}$. We consider special continuous linear functionals on $A^2(\Omega)$: the point evaluations. Let $f \in A^2(\Omega)$ and fix $z \in \Omega$. By Cauchy's integral theorem we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where $\gamma_s(t) = z + se^{it}$, $t \in [0, 2\pi], 0 < s \leq r$ and $D(z,r) = \{w : |w - z| < r\} \subset \Omega$. Using polar coordinates and integrating the above equality with respect to s between 0 and r we get

$$f(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w)\, d\lambda(w). \tag{1.1}$$

Then, by Cauchy–Schwarz,

$$\begin{aligned}
|f(z)| &\leq \frac{1}{\pi r^2} \int_{D(z,r)} 1 \,.\, |f(w)|\, d\lambda(w) \\
&\leq \frac{1}{\pi r^2} \left(\int_{D(z,r)} 1^2 \, d\lambda(w) \right)^{1/2} \left(\int_{D(z,r)} |f(w)|^2 \, d\lambda(w) \right)^{1/2} \\
&\leq \frac{1}{\pi^{1/2} r} \left(\int_\Omega |f(w)|^2 \, d\lambda(w) \right)^{1/2} \leq \frac{1}{\pi^{1/2} r} \|f\|.
\end{aligned}$$

If K is a compact subset of Ω, there is an $r(K) > 0$ such that for any $z \in K$ we have $D(z, r(K)) \subset \Omega$ and we get

$$\sup_{z\in K} |f(z)| \leq \frac{1}{\pi^{1/2} r(K)} \|f\|.$$

If $K \subset \Omega \subset \mathbb{C}^n$ we can find a polycylinder

$$P(z, r(K)) = \{w \in \mathbb{C}^n : |w_j - z_j| < r(K),\ j = 1, \dots, n\}$$

such that for any $z \in K$ we have $P(z, r(K)) \subset \Omega$. Hence by iterating the above Cauchy integrals we get

Proposition 1.1. *Let $K \subset \Omega$ be a compact set. Then there exists a constant $C(K)$, only depending on K such that*

$$\sup_{z\in K} |f(z)| \leq C(K)\, \|f\|, \tag{1.2}$$

for any $f \in A^2(\Omega)$.

Proposition 1.2. *$A^2(\Omega)$ is a Hilbert space.*

Proof. If $(f_k)_k$ is a Cauchy sequence in $A^2(\Omega)$, by (1.2), it is also a Cauchy sequence with respect to uniform convergence on compact subsets of Ω. Hence the sequence $(f_k)_k$ has a holomorphic limit f with respect to uniform convergence on compact subsets of Ω. On the other hand, the original L^2-Cauchy sequence has a subsequence, which converges pointwise almost everywhere to the L^2-limit of the original L^2-Cauchy sequence (see for instance [63]), and so the L^2-limit coincides with the holomorphic function f. Therefore $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and itself a Hilbert space. □

In the sequel we present basic facts about Hilbert spaces and their consequences for the Bergman spaces.

Proposition 1.3. *Let E be a nonempty, convex, closed subset of the Hilbert space H, i.e. for $x, y \in E$ one has $tx + (1-t)y \in E$, for each $t \in [0, 1]$. Then E contains a uniquely determined element of minimal norm.*

Proof. The parallelogram rule says that

$$\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2, \quad x, y \in H.$$

Let $\delta = \inf\{\|x\| : x \in E\}$. For $x, y \in E$ we have $\frac{1}{2}(x + y) \in E$, hence

$$1/4\,\|x - y\|^2 = 1/2\,\|x\|^2 + 1/2\,\|y\|^2 - \|1/2(x + y)\|^2,$$

implies that

$$\|x - y\|^2 \leq 2\,\|x\|^2 + 2\,\|y\|^2 - 4\delta^2.$$

So, if $\|x\| = \|y\| = \delta$, then $x = y$ (uniqueness).

By the definition of δ there exists a sequence $(y_k)_k$ in E such that $\|y_k\| \to \delta$ if $k \to \infty$. The estimate

$$\|y_k - y_m\|^2 \le 2\,\|y_k\|^2 + 2\,\|y_m\|^2 - 4\delta^2$$

implies that $(y_k)_k$ is a Cauchy sequence in H. Since H is complete there exists $x_0 \in H$ with $\|y_k - x_0\| \to 0$ and, as E is closed, we have $x_0 \in E$; the mapping $x \mapsto \|x\|$ is continuous and therefore $\|x_0\| = \lim_{k\to\infty} \|y_k\| = \delta$. □

Theorem 1.4. *Let M be a closed subspace of the Hilbert space H. Then there exist uniquely determined mappings*

$$P : H \longrightarrow M, \quad Q : H \longrightarrow M^{\perp}$$

such that
(1) $x = Px + Qx\,,\ \forall x \in H$
(2) *for $x \in M$ we have $Px = x$, hence $P^2 = P$ and $Qx = 0$; for $x \in M^{\perp}$ we have $Px = 0$, $Qx = x$, and $Q^2 = Q$.*
(3) *The distance of $x \in H$ to M is given by*

$$\inf\{\|x - y\| \ :\ y \in M\} = \|x - Px\|.$$

(4) *For each $x \in H$ we have*

$$\|x\|^2 = \|Px\|^2 + \|Qx\|^2.$$

(5) *P and Q are continuous, linear, self-adjoint operators.*
P and Q are the orthogonal projections of H onto M and $M^{\perp}$.

Proof. For each $x \in H$, the set $x + M = \{x + y \ :\ y \in M\}$ is convex. Hence, by Proposition 1.3, there exists a uniquely determined element of minimal norm in $x + M$, which is denoted by Qx. We set $Px = x - Qx$ and see that $Px \in M$, since $Qx \in x + M$.

Now we claim that $Qx \in M^{\perp}$. We have to show that$(Qx, y) = 0\,,\ \forall y \in M$: we can suppose that $\|y\| = 1$, then we have

$$(Qx, Qx) = \|Qx\|^2 \le \|Qx - \alpha y\|^2 = (Qx - \alpha y, Qx - \alpha y), \quad \forall \alpha \in \mathbb{C}$$

by the minimality of Qx. Therefore we get

$$0 \le -\alpha(y, Qx) - \overline{\alpha}(Qx, y) + |\alpha|^2,$$

setting $\alpha = (Qx, y)$, we obtain $0 \le -|(Qx, y)|^2$ and $(Qx, y) = 0$; hence $Q : H \longrightarrow M^{\perp}$.

If $x = x_0 + x_1$ with $x_0 \in M$ and $x_1 \in M^{\perp}$, then $x_0 - Px = Qx - x_1$, and since $M \cap M^{\perp} = \{0\}$ we obtain $x_0 = Px$ and $x_1 = Qx$, therefore P and Q are uniquely determined. In a similar way, we get that

$$P(\alpha x + \beta y) - \alpha Px - \beta Py = \alpha Qx + \beta Qy - Q(\alpha x + \beta y).$$

The left side belongs to M, the right side belongs to $M^\perp$, hence both sides are 0, which proves that P and Q are linear.

Property 3 follows by the definition of Q, property 4 by the fact that

$$(Px, Qx) = 0, \quad \forall x \in H.$$

In addition we have

$$\|Q(x-y)\| = \inf\{\|x - y + m\| \ : \ m \in M\} \leq \|x - y\|,$$

hence Q and $P = I - Q$ are continuous.

For $x, y \in H$ we have

$$(Px, y) = (Px, Py + Qy) = (Px, Py) \quad \text{and} \quad (x, Py) = (Px + Qx, Py) = (Px, Py)$$

hence $(Px, y) = (x, Py)$, and so P is self-adjoint. □

Corollary 1.5. *If $M \neq H$ is a closed, proper subspace of the Hilbert space H, then there exists an element $y \neq 0$ with $y \perp M$.*

Proof. Let $x \in H$ such that $x \notin M$. Set $y = Qx$: then $x \neq Px$ implies $y \neq 0$. □

The next result is the Riesz representation theorem.

Theorem 1.6. *Let L be a continuous linear functional on the Hilbert space H. Then there exists a uniquely determined element $y \in H$ such that $Lx = (x, y)$, $\forall x \in H$.*

Proof. If $L(x) = 0$, $\forall x \in H$, then we set $y = 0$. Otherwise we define $M = \{x \ : \ Lx = 0\}$. Then, by the continuity of L, the subspace M of H is closed. By Corollary 1.5 we have $M^\perp \neq 0$. Let $z \in M^\perp$ with $z \neq 0$. Then $Lz \neq 0$. Now set $y = \alpha z$, where $\overline{\alpha} = \frac{Lz}{\|z\|^2}$. Then $y \in M^\perp$ and

$$Ly = L(\alpha z) = \frac{\overline{Lz}}{\|z\|^2} Lz = \frac{|Lz|^2}{\|z\|^2} = (y, y) = |\alpha|^2 (z, z).$$

For $x \in H$ we define

$$x' = x - \frac{Lx}{(y,y)} y \quad \text{and} \quad x'' = \frac{Lx}{(y,y)} y.$$

Then we obtain $Lx' = 0$ and $x' \in M$, hence $(x', y) = 0$ and

$$(x, y) = (x'', y) = \left(\frac{Lx}{(y,y)} y, y \right) = Lx.$$

If $(x, y) = (x, y')$, $\forall x \in H$, then we get $(x, y - y') = 0$, $\forall x \in H$, in particular $(y - y', y - y') = 0$. Therefore $y = y'$, which shows that y is uniquely determined. □

Corollary 1.7. *Let H be a Hilbert space and $L \in H'$ a continuous linear functional. Then the dual norm*

$$\|L\| = \sup\{|Lx| : \|x\| \leq 1\}$$