Lectures in Mathematics **ETH** Zürich

Jürgen Moser

Selected Chapters in the Calculus of Variations

Lecture Notes by Oliver Knill

Birkhäuser

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These lecture notes describe the Aubry-Mather-Theory within the calculus of variations. The text consists of the translated original lectures of Jürgen Moser and a bibliographic appendix with comments on the current state-of-the-art in this field of interest. Students will find a rapid introduction to the calculus of variations, leading to modern dynamical systems theory. Differential geometric applications are discussed, in particular billiards and minimal geodesics on the two-dimensional torus. Many exercises and open questions make this book a valuable resource for both teaching and research.

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0.1 Introduction

These lecture notes describe a new development in the calculus of variations which is called **Aubry–Mather–Theory**.

The starting point for the theoretical physicist Aubry was a model for the description of the motion of electrons in a two-dimensional crystal. Aubry investigated a related discrete variational problem and the corresponding minimal solutions.

On the other hand, Mather started with a specific class of area-preserving annulus mappings, the so-called **monotone twist maps**. These maps appear in mechanics as Poincaré maps. Such maps were studied by Birkhoff during the 1920s in several papers. In 1982, Mather succeeded to make essential progress in this field and to prove the existence of a class of closed invariant subsets which are now called **Mather sets**. His existence theorem is based again on a variational principle.

Although these two investigations have different motivations, they are closely related and have the same mathematical foundation. We will not follow those approaches but will make a connection to classical results of Jacobi, Legendre, Weierstrass and others from the 19th century.

Therefore in Chapter I, we will put together the results of the classical theory which are the most important for us. The notion of **extremal fields** will be most relevant.

In Chapter II we will investigate variational problems on the 2-dimensional torus. We will look at the corresponding global minimals as well as at the relation between minimals and extremal fields. In this way, we will be led to Mather sets.

Finally, in Chapter III, we will learn the connection with monotone twist maps, the starting point for Mather's theory. In this way we will arrive at a discrete variational problem which forms the basis for Aubry's investigations.

This theory has additional interesting applications in differential geometry. One of those is the geodesic flow on two-dimensional surfaces, especially on the torus. In this context the **minimal geodesics** play a distinguished role. They were investigated by Morse and Hedlund in 1932.

As Bangert has shown, the theories of Aubry and Mather lead to new results for the geodesic flow on the two-dimensional torus. As the last section of these lecture notes will show, the restriction to two dimensions is essential. These differential geometric questions are treated at the end of the third chapter.

The beautiful survey article of Bangert should be at hand when reading these lecture notes.

Our description aims less at generality. We rather aim to show the relations of newer developments with classical notions like extremal fields. Mather sets will appear as 'generalized extremal fields' in this terminology.

For the production of these lecture notes I was assisted by O. Knill to whom I want to express my thanks.

Zürich, September 1988, J. Moser

0.2 On these lecture notes

These lectures were presented by J. Moser in the spring of 1988 at the Eidgenössische Technische Hochschule (ETH) Zürich. Most of the students were enrolled in the 6th to the 8th semester of the 4 year Mathematics curriculum. There were also graduate students and visitors from the research institute at the ETH (FIM) in the auditorium.

In the last decade, the research on this particular topic of the calculus of variations has made some progress. A few hints to the literature are listed in an Appendix. Because some important questions are still open, these lecture notes are maybe of more than historical value.

The notes were typed in the summer of 1988. J. Moser had looked carefully through the notes in September 1988. Because the text editor in which the lecture were originally written is now obsolete, the typesetting was done from scratch with LaTeX in the year 2000. The original had not been changed except for small, mostly stylistic or typographical corrections. In 2002, an English translation was finished and figures were added.

Cambridge, MA, December 2002, O. Knill

Chapter 1

One-dimensional variational problems

1.1 Regularity of the minimals

Let Ω be an open region in $\mathbb{R}^{n+1}$. We assume that Ω is simply connected. A point in Ω has the coordinates $(t, x_1, ..., x_n) = (t, x)$. Let $F = F(t, x, p) \in C^r(\Omega \times \mathbb{R}^n)$ with $r \geq 2$ and let (t_1, a) and (t_2, b) be two points in Ω. The space

$$\Gamma := \{\gamma : t \to x(t) \in \Omega \mid x \in C^1[t_1, t_2],\ x(t_1) = a, x(t_2) = b\ \}$$

consists of all continuously differentiable curves which start at (t_1, a) and end at (t_2, b). On Γ is defined the functional

$$I(\gamma) = \int_{t_1}^{t_2} F(t, x(t), \dot{x}(t))\, dt\ .$$

Definition. We say that $\gamma^* \in \Gamma$ is **minimal** in Γ if

$$I(\gamma) \geq I(\gamma^*),\ \forall \gamma \in \Gamma\ .$$

We first search for necessary conditions for a minimum of I while assuming the existence of a minimal.

Remark. A minimum does not need to exist in general:

- It is possible that $\Gamma = \emptyset$.
- It is also possible that a minimal γ^* is contained only in $\overline{\Omega}$.

- Finally, the infimum could exist without the minimum being achieved.

Example. Let $n = 1$ and $F(t, x, \dot{x}) = t^2 \cdot \dot{x}^2, (t_1, a) = (0, 0), (t_2, b) = (1, 1)$. We have

$$\gamma_m(t) = t^m,\ I(\gamma_m) = \frac{1}{m+3},\ \inf_{m \in \mathbb{N}} I(\gamma_m) = 0,$$

but for all $\gamma \in \Gamma$ one has $I(\gamma) > 0$.

Theorem 1.1.1. *If γ^* is minimal in Γ, then*

$$F_{p_j}(t, x^*, \dot{x}^*) = \int_{t_1}^{t} F_{x_j}(s, x^*, \dot{x}^*)\, ds = const$$

for all $t_1 \leq t \leq t_2$ and $j = 1, ..., n$. These equations are called **integrated Euler equations**.

Definition. One calls γ^* **regular** if $\det(F_{p_i p_j}) \neq 0$ for $x = x^*, p = \dot{x}^*$.

Theorem 1.1.2. *If γ^* is a regular minimal, then $x^* \in C^2[t_1, t_2]$ and one has for $j = 1, \ldots, n$,*

$$\frac{d}{dt} F_{p_j}(t, x^*, \dot{x}^*) = F_{x_j}(t, x^*, \dot{x}^*) \tag{1.1}$$

These equations are called **Euler equations**.

Definition. An element $\gamma^* \in \Gamma$ satisfying the Euler equations (1.1) is called an **extremal** in Γ.

Warning. Not every extremal solution is a minimal!

Proof of Theorem 1.1.1. We assume that γ^* is minimal in Γ. Let $\xi \in C_0^1(t_1, t_2) = \{x \in C^1[t_1, t_2] \mid x(t_1) = x(t_2) = 0\}$ and $\gamma_\epsilon : t \mapsto x(t) + \epsilon\xi(t)$. Because Ω is open and $\gamma \in \Omega$, also $\gamma_\epsilon \in \Omega$ for small enough ϵ. Therefore,

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(F_{p_j}(s)\dot{\xi}_j + F_{x_j}(s)\xi_j \right) ds \\ &= \int_{t_1}^{t_2} (\lambda(t), \dot{\xi}(t))\, dt \end{aligned}$$

with $\lambda_j(t) = F_{p_j}(t) - \int_{t_1}^{t_2} F_{x_j}(s)\, ds$. Theorem 1.1.1 is now a consequence of the following lemma. □

Lemma 1.1.3. *If* $\lambda \in C[t_1, t_2]$ *and*

$$\int_{t_1}^{t_2} (\lambda, \dot{\xi})\ dt = 0, \quad \forall \xi \in C_0^1[t_1, t_2]$$

then $\lambda = const.$

Proof. Define $c = (t_2 - t_1)^{-1} \int_{t_1}^{t_2} \lambda(t)\ dt$ and put $\xi(t) = \int_{t_1}^{t} (\lambda(s) - c)\ ds$. Now $\xi \in C_0^1[t_1, t_2]$. By assumption we have:

$$0 = \int_{t_1}^{t_2} (\lambda, \dot{\xi})\ dt = \int_{t_1}^{t_2} (\lambda, (\lambda - c))\ dt = \int_{t_1}^{t_2} (\lambda - c)^2\ dt\ ,$$

where the last equation followed from $\int_{t_1}^{t_2} (\lambda - c)\ dt = 0$. Because λ is continuous, this implies with $\int_{t_1}^{t_2} (\lambda - c)^2\ dt = 0$ the claim $\lambda = const.$ □

Proof of Theorem 1.1.2. Put $y_j^* = F_{p_j}(t, x^*, p^*)$. Since by assumption $\det(F_{p_i p_j}) \neq 0$ at every point $(t, x^*(t), \dot{x}^*(t))$, the implicit function theorem assures that functions $p_k^* = \phi_k(t, x^*, y^*)$ exist, which are locally C^1. From Theorem 1.1.1 we know

$$y_j^* = const - \int_{t_1}^{t} F_{x_j}(s, x^*, \dot{x}^*)\ ds \in C^1 \tag{1.2}$$

and so

$$\dot{x}_k^* = \phi_k(t, x^*, y^*) \in C^1\ .$$

Therefore $x_k^* \in C^2$. The Euler equations are obtained from the integrated Euler equations in Theorem 1.1.1. □

Theorem 1.1.4. *If* γ^* *is minimal, then*

$$(F_{pp}(t, x^*, y^*)\zeta, \zeta) = \sum_{i,j=1}^{n} F_{p_i p_j}(t, x^*, y^*)\zeta_i \zeta_j \geq 0$$

holds for all $t_1 < t < t_2$ *and all* $\zeta \in \mathbb{R}^n$.

Proof. Let γ_ϵ be defined as in the proof of Theorem 1.1.1. Then $\gamma_\epsilon : t \mapsto x^*(t) + \epsilon\xi(t)$, $\xi \in C_0^1$.

$$0 \leq II \quad := \quad \frac{d^2}{(d\epsilon)^2} I(\gamma_\epsilon)|_{\epsilon=0} \tag{1.3}$$

$$= \int_{t_1}^{t_2} (F_{pp}\dot{\xi}, \dot{\xi}) + 2(F_{px}\dot{\xi}, \xi) + (F_{xx}\xi, \xi)\ dt\ . \tag{1.4}$$

II is called the **second variation** of the functional I. Let $t \in (t_1, t_2)$ be arbitrary. We construct now special functions $\xi_j \in C_0^1(t_1, t_2)$:

$$\xi_j(t) = \zeta_j \psi(\frac{t-\tau}{\epsilon}) ,$$

where $\zeta_j \in \mathbb{R}$ and $\psi \in C^1(\mathbb{R})$ by assumption, $\psi(\lambda) = 0$ for $|\lambda| > 1$ and $\int_{\mathbb{R}} (\psi')^2 \, d\lambda = 1$. Here ψ' denotes the derivative with respect to the new time variable τ, which is related to t as follows:

$$t = \tau + \epsilon\lambda, \ \epsilon^{-1} dt = d\lambda .$$

The equations

$$\dot{\xi}_j(t) = \epsilon^{-1} \zeta_j \psi'(\frac{t-\tau}{\epsilon})$$

and (1.3) give

$$0 \leq \epsilon^3 II = \int_{\mathbb{R}} (F_{pp}\zeta, \zeta)(\psi')^2(\lambda) \, d\lambda + O(\epsilon) .$$

For $\epsilon > 0$ and $\epsilon \to 0$ this means that

$$(F_{pp}(t, x(t), \dot{x}(t))\zeta, \zeta) \geq 0 .$$

□

Definition. We call the function F **autonomous**, if F is independent of t.

Theorem 1.1.5. *If F is autonomous, every regular extremal solution satisfies*

$$H = -F + \sum_{j=1}^{n} p_j F_{p_j} = const .$$

The function H is also called the **energy**. *In the autonomous case we have therefore energy conservation.*

Proof. Because the partial derivative H_t vanishes, one has

$$\begin{aligned} \frac{d}{dt} H &= \frac{d}{dt}(-F + \sum_{j=1}^{n} p_j F_{p_j}) \\ &= \sum_{j=1}^{n} (-F_{x_j} \dot{x}_j - F_{p_j} \ddot{x}_j + \ddot{x}_j F_{p_j} + \dot{x}_j \frac{d}{dt} F_{p_j}) \\ &= \sum_{j=1}^{n} -F_{x_j} \dot{x}_j - F_{p_j} \ddot{x}_j + \ddot{x}_j F_{p_j} + \dot{x}_j F_{x_j} = 0 . \end{aligned}$$

Because the extremal solution was assumed to be regular, we could use the Euler equations (Theorem 1.1.2) in the last step. □

In order to obtain sharper regularity results we change the variational space. We have seen that if F_{pp} is not degenerate, then $\gamma^* \in \Gamma$ is two times differentiable even though the elements in Γ are only C^1. This was the statement of the regularity Theorem 1.1.2.

We consider now a bigger class of curves

$$\Lambda = \{\gamma : [t_1, t_2] \to \Omega,\ t \mapsto x(t), x \in \mathrm{Lip}[t_1, t_2], x(t_1) = a, x(t_2) = b\ \} .$$

$\mathrm{Lip}[t_1, t_2]$ denotes the space of Lipschitz continuous functions on the interval $[t_1, t_2]$. Note that $\dot{x}$ is now only measurable and bounded. Nevertheless there are results analogous to Theorem 1.1.1 or Theorem 1.1.2:

Theorem 1.1.6. *If γ^* is a minimal in Λ, then*

$$F_{p_j}(t, x^*, \dot{x}^*) - \int_{t_1}^{t_2} F_{x_j}(s, x^*, \dot{x}^*)\, ds = const \tag{1.5}$$

for Lebesgue almost all $t \in [t_1, t_2]$ and all $j = 1, ..., n$.

Proof. As in the proof of Theorem 1.1.1 we put $\gamma_\epsilon = \gamma + \epsilon\xi$, but this time, ξ is in

$$\mathrm{Lip}_0[t_1, t_2] := \{\gamma : t \mapsto x(t) \in \Omega,\ x \in \mathrm{Lip}[t_1, t_2], x(t_1) = x(t_2) = 0\ \} .$$

So,

$$\begin{aligned} 0 &= \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} \\ &= \lim_{\epsilon \to 0} (I(\gamma_\epsilon) - I(\gamma_0))/\epsilon \\ &= \lim_{\epsilon \to 0} \int_{t_1}^{t_2} [F(t, \gamma^* + \epsilon\xi, \dot{\gamma}^* + \epsilon\dot{\xi}) - F(t, \gamma^*, \dot{\gamma}^*)]/\epsilon\ dt . \end{aligned}$$

To take the limit $\epsilon \to 0$ inside the integral, we use Lebesgue's dominated convergence theorem: for fixed t we have

$$\lim_{\epsilon \to 0} [F(t, \gamma^* + \epsilon\xi, \dot{\gamma^*} + \epsilon\dot{\xi}) - F(t, \gamma^*, \dot{\gamma^*})]/\epsilon = (F_x, \xi) + (F_p, \dot{\xi})$$

and

$$\frac{F(t, \gamma^* + \epsilon\xi, \dot{\gamma}^* + \epsilon\dot{\xi}) - F(t, \gamma, \dot{\gamma})}{\epsilon} \leq \sup_{s \in [t_1, t_2]} |F_x(s, x(s), \dot{x}(s)|\xi(s) + |F_p(s, x(s)|\dot{\xi}(s) .$$

The last expression is in $L^1[t_1, t_2]$. Applying Lebesgue's theorem gives

$$0 = \frac{d}{d\epsilon} I(\gamma_\epsilon)|_{\epsilon=0} = \int_{t_1}^{t_2} (F_x, \xi) + (F_p, \dot{\xi})\ dt = \int_{t_1}^{t_2} \lambda(t)\dot{\xi}\ dt$$

with $\lambda(t) = F_p - \int_{t_1}^{t_2} F_x\ ds$. This is bounded and measurable.

Define $c = (t_2 - t_1)^{-1} \int_{t_1}^{t} \lambda(t)\, dt$ and put $\xi(t) = \int_{t_1}^{t_2} (\lambda(s) - c)\, ds$. We get $\xi \in \mathrm{Lip}_0[t_1, t_2]$ and in the same way as in the proof of Theorem 1.1.4 or Lemma 1.1.3 one concludes

$$0 = \int_{t_1}^{t_2} (\lambda, \dot{\xi})\, dt = \int_{t_1}^{t_2} (\lambda, (\lambda(t) - c)))\, dt = \int_{t_1}^{t_2} (\lambda - c)^2\, dt\ ,$$

where the last equation followed from $\int_{t_1}^{t_2} (\lambda - c)\, dt = 0$. This means that $\lambda = c$ for almost all $t \in [t_1, t_2]$. □

Theorem 1.1.7. *If γ^* is a minimal in Λ and $F_{pp}(t, x, p)$ is positive definite for all $(t, x, p) \in \Omega \times \mathbb{R}^n$, then $x^* \in C^2[t_1, t_2]$ and*

$$\frac{d}{dt} F_{p_j}(t, x^*, \dot{x}^*) = F_{x_j}(t, x^*, \dot{x}^*)$$

for $j = 1, ..., n$.

Proof. The proof uses the integrated Euler equations in Theorem 1.1.1. It makes use of the fact that a solution of the implicit equation $y = F_p(t, x, p)$ for $p = \Phi(t, x, y)$ is **globally unique**. Indeed: if two solutions p and q would exist with

$$y = F_p(t, x, p) = F_q(t, x, q)\ ,$$

it would imply that

$$0 = (F_p(t, x, p) - F_p(t, x, q), p - q) = (A(p - q), p - q)$$

with

$$A = \int_0^1 F_{pp}(t, x, p + \lambda(q - p))\, d\lambda$$

and because A was assumed to be positive definite, $p = q$ follows.

From the integrated Euler equations we know that

$$y(t) = F_p(t, x, \dot{x})$$

is continuous with bounded derivatives. Therefore $\dot{x} = \Phi(t, x, y)$ is absolutely continuous. Integration leads to $x \in C^1$. The integrable Euler equations of Theorem 1.1.1 tell now that F_p is even in C^1 and we get, with the already proven global uniqueness result, that $\dot{x}$ is in C^1 and hence that x is in C^2. We obtain the Euler equations by differentiating (1.5). □