

DISCRETE MATHEMATICS AND ITS APPLICATIONS

# INTRODUCTION TO ENUMERATIVE AND ANALYTIC COMBINATORICS SECOND EDITION

**Miklós Bóna**

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*To Linda*  
*To Mikike, Benny, and Vinnie*





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## *Foreword to the first edition*

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What could be a more basic mathematical activity than counting the number of elements of a finite set? The misleading simplicity that defines the subject of enumerative combinatorics is in fact one of its principal charms. Who would suspect the wealth of ingenuity and of sophisticated techniques that can be brought to bear on a such an apparently superficial endeavor? Miklós Bóna has done a masterful job of bringing an overview of all of enumerative combinatorics within reach of undergraduates. The two fundamental themes of bijective proofs and generating functions, together with their intimate connections, recur constantly. A wide selection of topics, including several never appearing before in a textbook, are included that give an idea of the vast range of enumerative combinatorics. In particular, for those with sufficient background in undergraduate linear algebra and abstract algebra there are many tantalizing hints of the fruitful connection between enumerative combinatorics and algebra that plays a central role in the subject of algebraic combinatorics. In a foreword to another book by Miklós Bóna I wrote, “This book can be utilized at a variety of levels, from random samplings of the treasures therein to a comprehensive attempt to master all the material and solve all the exercises. In whatever direction the reader’s tastes lead, a thorough enjoyment and appreciation of a beautiful area of combinatorics is certain to ensue.” Exactly the same sentiment applies to the present book, as the reader will soon discover.

Richard Stanley  
Cambridge, Massachusetts  
June 2005



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## *Preface to the second edition*

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There are at least three ways to use this book. If one decides to cover every chapter, and some of the exercises containing new material, then one can teach a two-semester combinatorics course from the book. Instructors looking for a text for a course in enumerative combinatorics can teach such a course selecting chapters with the strongest focus on counting, such as Chapters 2, 3, 4, 5, 7, and 9. Finally, one can teach a one-semester course with a not-quite-as-strong focus on enumeration by using Chapters 1, 2, 3, and then the desired chapters from the rest of the book.

Our hope is that our book can broaden access to the fascinating topics of enumerative and analytic combinatorics, and will prepare readers for the more advanced, classic books of the field, such as *Enumerative Combinatorics* by Richard Stanley and *Analytic Combinatorics* by Philippe Flajolet and Robert Sedgewick.

This current edition of the book contains a new chapter on analytic combinatorics. Instructors trying to teach that topic to a relatively novice audience often hit a roadblock when they realize that the audience is unfamiliar with complex analysis. The goal of this chapter is to at least partially overcome that roadblock by showing the reader some of the powerful techniques of that field that do *not* require the residue theorem or complex integration. Hopefully, readers will find the results interesting and that will entice them to learn the techniques that we only mention here. Strengthening the analytic flavor of the book, Chapters 4 and 5 have been enhanced by new sections discussing advanced applications of generating functions. Finally, we added new exercises to all chapters. Just as before, numerous exercises contain material not discussed in the text, which allows instructors to extend the time they spend on a given topic.

Combinatorics is a rapidly expanding field, and we hope that our book will increase the number of students with an interest in it even further.

Gainesville, FL  
August 2015



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## *Acknowledgments*

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First, I am thankful to my grandfather, the late Dr. János Bóna, who introduced me to counting problems at an early age. I am indebted to the researchers whose work attracted me to enumerative combinatorics. They include Richard Stanley, Herbert Wilf, Bruce Sagan, Catherine Yan, Francesco Brenti, and many others. The second edition of the book has an analytic flavor that would not have been possible if I had not learned analytic combinatorics from, and been influenced by, Philippe Flajolet, Robert Sedgewick, Boris Pittel, Robin Pemantle, and E. Rodney Canfield.

As always, my gratitude is extended to my wife Linda and my sons Miki, Benny, and Vinnie, who tolerated another book project reasonably well.



## Frequently used notation

- $A(n, k)$  number of  $n$ -permutations with  $k$  ascending runs
- $B(n)$  number of all partitions of an  $n$ -element set
- $b(n, k)$  number of  $n$ -permutations with  $k$  inversions
- $c(n, k)$  number of  $n$ -permutations with  $k$  cycles
- $c_n$  the Catalan number  $c_n = \binom{2n}{n}/(n+1)$
- $C_n$  a cycle on  $n$  vertices
- $|\mathcal{F}|$  number of edges of the hypergraph  $\mathcal{F}$
- $G_i$  stabilizer of the element  $i$  under the action of the group  $G$
- $G - v$  graph  $G$  with its vertex  $v$  and all edges adjacent to  $v$  removed
- $i(p)$  number of inversions of the permutation  $p$
- $i^G$  orbit of the element  $i$  under the action of the group  $G$
- $K_n$  complete graph on  $n$  vertices
- $K_{m,n}$  complete bipartite graph with  $m$  red and  $n$  blue vertices
- $\chi_G$  chromatic polynomial of the graph  $G$
- $\chi(G)$  chromatic number of the graph  $G$
- $n!$   $n(n-1) \cdots 1$
- $\binom{n}{k}$   $\frac{(n)_k}{k!}$
- $[n]$  set  $\{1, 2, \dots, n\}$
- $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  Gaussian polynomial  $\prod_{i=1}^k \frac{q^n - q^{i-1}}{q^k - q^{i-1}}$
- $(n)_m$   $n(n-1) \cdots (n-m+1)$
- $p(n)$  number of partitions of the integer  $n$
- $p_k(n)$  number of partitions of the integer  $n$  into at most  $k$  parts



- $|S|$  number of elements of the set  $S$
- $S_n$  set of all  $n$ -permutations
- $S_n(q)$  number of all  $n$ -permutations avoiding the pattern  $q$
- $S(n, k)$  number of partitions of the set  $[n]$  into  $k$  blocks
- $s(n, k) = (-1)^{n-k} c(n, k)$