A Study on Eigenvalues of Higher-Order Tensors and Related Polynomial Optimization Problems

（高阶张量特征值和相关多项式优化问题研究）

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To the first author's parents:
Yang Cunliang and Huang Linlin
And wife:
Chen Yiying

Preface

In this book, we study the eigenvalues problems of higher order tensors and several specific polynomial optimization problems.

The concepts of eigenvalues and eigenvectors of higher order tensors were presented in 2005 by Qi and Lim independently. Since then, more and more scholars have devoted to this new research field and have made great progress in the past decade. Particularly the various properties of H-eigenvalues of nonnegative tensors and solving methods have been developed. In the first part of this book we systematically introduce related contents. Problems on finding the maximal or minimal H-eigenvalues and Z-eigenvalues are special class of polynomial optimization problems, while polynomial optimization is attracting great attention due to its importance and has been witnessed rapid development in the past years. It is well known that in general a polynomial optimization problem is nonconvex and NP-hard. So people attempt to use suitable relaxed convex optimization problem to approximate a concrete nonconvex optimization problem. Semidefinite programming (SDP) is an important branch of convex optimization and has been studied extensively in theory as well as in algorithms in the past two decades. Besides it is important in itself right, SDP has been found to be very useful as the convex approximation of nonconvex quadratic programming and some discrete optimization problems. In the second part of this book we devote to studying several special polynomial optimization problems by using SDP relaxation strategy, such as maximal real tensor eigenvalue problem and their extended problems.

Most of the contents of this book are composed of PhD dissertation of the first author, which have been published in journals.

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Yang Qingzhi
Yang Yuning
Nov. 2014

Notations

$x, y, \cdots$	Vectors
$A, B, \cdots$	Matrices
$\mathcal{A}, \mathcal{B}, \cdots$	Tensors
$\mathbb{R}, \mathbb{C}, \mathbb{Q}$	Real, complex and rational number field
$\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_{++}$	n-dimensional real (nonnegative, positive) vector space
$\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}_+, \mathbb{R}^{n\times n}_{++}$	$n \times n$ dimensional real (nonnegative, positive) matrix space
$S^{n\times n}, S^{n\times n}_+, S^{n\times n}_{++}$	$n \times n$ real symmetric (positive semidefinite, positive definite) matrix space
$\|\cdot\|$	ℓ_2 norm of a vector
$\|\cdot\|_F$	Frobenius norm of a matrix or a tensor
T	Vector or matrix transpose
rank$(\cdot)$	Rank of a matrix
trace$(\cdot)$	Trace of a marix or a tensor
$\bullet$	Matrix inner product
$\circ$	Hadamard operator
$\otimes$	Kronecker product
$\times_d$	Tensor-matrix mode-d product
$\|\cdot\|$	Entry-wise absolute values of a vector, a matrix or a tensor
diag(X)	The diagonal entries of a matrix X
Diag(x)	A matrix whose diagonal entries are the vector x
$\boldsymbol{M}(\cdot)$	Tensor-matrix unfolding operator
$\boldsymbol{T}(\cdot)$	Matrix-tensor folding operator
$\boldsymbol{V}(\cdot)$	Matrix-vector unfolding operator
$\boldsymbol{L}(\cdot)$	Vector-matrix folding operator

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Chapter 1
Introduction

Tensor is a hot topic in the past decade. Nowadays, many real world problems can be modeled as tensor problems, just to name a few: signal processing[36, 101], data analysis[17, 32], chemometrics[12, 13, 111], hypergraph theory[34, 66], diffusion magnetic resonance imaging (MRI) [3, 6, 44], quantum entanglement in quantum physics [35], higher order Markov chains[107] and elastic materials analysis[50, 77]. Specifically, a tensor can be viewed as a multiarray: if a vector $a = [a_1, \cdots, a_n]^{\mathrm{T}} \in \mathbb{R}^n$ is a one-way-array, a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a two-way-array, then a tensor of order m dimension n: $\mathcal{A} = (a_{i_1\cdots i_n})$ is a multiarray. Here "order" refers to the number of indices of each entry of $\mathcal{A}$, e.g., a matrix is a tensor of order 2. Just imagine we have l matrices of size $m \times n$. By stacking them up together, we get a cube, which is a $l \times m \times n$ tensor. If we merge k such tensors together, then we get a hyper-cube, which is a $k \times l \times m \times n$ tensor, and so on. Particularly, if $k = l = m = n = \cdots$, i.e., all the dimensions of a tensor are the same, then the tensor is called a "square" tensor, which generalizes the "square" matrix and is an important type of tensors studied in this book.

We are particularly interested in the (spectral) properties of eigenvalue problems of higher order tensor and related polynomial optimization problems. In this book, the main concerns are the following five topics:

- Spectral properties and algorithms of H-eigenvalue problems;
- Spectral properties and algorithms of H-singular value problems;
- Properties and algorithms of Z-eigenvalue problems;
- Approximation methods of biquadratic optimization problems;
- Approximation methods of trilinear optimization problems.

We will introduce the related concepts and problems in the rest of this chapter.

1.1 Eigenvalues problems of higher order tensors

It is well known that eigenvalues play an important role in matrix theory. For a matrix $A \in \mathbb{C}^{n\times n}$, if there exists a pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x \neq 0$ such that

$$Ax = \lambda x,$$

then λ is called an eigenvalue of A, and x is an eigenvector corresponding to λ. To study tensor problems, one may naturally ask a question: can we define eigenvalues and eigenvectors on tensors? The answer is positive. In 2005, the concept of eigenvalues and eigenvectors of a symmetric tensor with order even was introduced by Qi[119].

To be more specific, let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order m n-th dimensional real square tensor. If there is a complex number λ and a nonzero complex vector x that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

then λ is called an eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ. In problem (1.1), $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose i-th entries are given by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m = 1}^{n} a_{i, i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},$$
$$(x^{[m-1]})_i = x_i^{m-1},$$

respectively. If λ and x are restricted in the real field, then (λ, x) is called an H-eigenpaire. If an eigenvalue is not an H-eigenvalue, we call it an N-eigenvalue of $\mathcal{A}$. Besides the H-eigenvalues, Qi defined the Z-eigenvalues[119]: a real number λ and a real vector x are called Z-eigenvalue of $\mathcal{A}$ and a Z-eigenvector of $\mathcal{A}$ associated with the Z-eigenvalue λ respectively, if they are solutions of the following system:

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ \|x\| = 1. \end{cases} \tag{1.2}$$

Here $\|\cdot\|$ is the Euclidean norm. If λ and x are complex, then they are called E-eiganvalue and E-eigenvector.

In the same year, Lim[94] independently defined eigenvalues for general real tensors in the real field. In his work, the l^k eigenvalues are H-eigenvalues, while the l^2 eigenvalues of tensors are the Z-eigenvalues. Note that in the case of $m = 2$, both the H-eigenvalue and Z-eigenvalue collapse to the eigenvalues of a matrix.

The concept of singular values and singular vectors can be generalized to higher order tensors as well. Let us recall in the real case that for a matrix $A \in \mathbb{R}^{n_1 \times n_2}$, if there exists a triple $(\lambda, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$Ax_2 = \lambda x_1 \text{ and } A^{\mathrm{T}} x_1 = \lambda x_2,$$

then λ is called a singular value of A, and x_1 and x_2 are the left and right singular vector corresponding to λ, respectively.

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