CALCULUS
Theory and Applications

Volume 2

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CALCULUS

Theory and Applications

Volume 2

Preface

This volume is mainly devoted to multivariable calculus. It is generally easier to consider linear functions than nonlinear ones. Therefore, the linear functions are presented first in terms of basic linear algebra. Then this is used to unify the presentation of nonlinear functions. All theorems are proved although the most difficult ones are in the appendices along with some other applications such as curvilinear coordinates.

Like Volume I, really difficult theoretical sections have a dragon at the beginning. These sections are optional and are there for anyone who is interested. It seems to me that a math book should provide explanations of the theorems, but these explanations can be skipped if there is no interest.

Supplementary material for this text including routine exercise sets is available at http://www.math.byu.edu/~klkuttle/CalculusMaterials.

I am grateful to World Scientific for publishing this volume and also to Kate Phillips, and Maple for drawing the pictures.

Contents

Chapter 1

Matrices And Linear Transformations

1.1 Matrix Arithmetic

1.1.1 *Addition And Scalar Multiplication Of Matrices*

Numbers are also called **scalars**. In this book, scalars will be real numbers or complex numbers.

A **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{pmatrix}$$

The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. The first row is $(1\ 2\ 3\ 4)$, the second row is $(5\ 2\ 8\ 7)$ and so forth. The first column is $\begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$. When specifying the size of a matrix, you always list the number of rows before the number of columns. Also, you can remember the columns are like columns in a Greek temple. They stand upright while the rows just lay there like rows made by a tractor in a plowed field. Entries of the matrix are identified according to position in the matrix. For example, 8 is in position $2,3$ because it is in the second row and the third column. You might remember that you always list the rows before the columns by using the phrase **Row**man **C**atholic.

The symbol (a_{ij}) refers to a matrix. The entry in the i^{th} row and the j^{th} column of this matrix is denoted by a_{ij}. Using this notation on the above matrix, $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

There are various operations which are done on matrices. Matrices can be added, multiplied by a scalar, and multiplied by other matrices. To illustrate scalar multiplication, consider the following example in which a matrix is being multiplied by the scalar 3.

$$3\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{pmatrix}.$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar. If A is an $m \times n$ matrix, $-A$ equals $(-1)A$ because $A + (-1)A = 0$.

Two matrices must be the same size to be added. The sum of two matrices is a matrix which is obtained by adding the corresponding entries. Thus

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 2 & 8 \\ 6 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 5 & 12 \\ 11 & -2 \end{pmatrix}.$$

Two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

because they are different sizes. As noted above, you write (c_{ij}) for the matrix C whose ij^{th} entry is c_{ij}. In doing arithmetic with matrices you must define what happens in terms of the c_{ij} sometimes called the **entries** of the matrix or the **components** of the matrix.

The above discussion stated for general matrices is given in the following definition.

Definition 1.1. (Scalar Multiplication) If $A = (a_{ij})$ and k is a scalar, then $kA = (ka_{ij})$.

Example 1.1. $7\begin{pmatrix} 2 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 14 & 0 \\ 7 & -28 \end{pmatrix}.$

Definition 1.2. (Addition) If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices. Then $A + B = C$ where

$$C = (c_{ij})$$

for $c_{ij} = a_{ij} + b_{ij}$.

Example 1.2.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{pmatrix}.$$

To save on notation, we will often use A_{ij} to refer to the ij^{th} entry of the matrix A.

Definition 1.3. (The **zero matrix**) The $m \times n$ zero matrix is the $m \times n$ matrix having every entry equal to zero. It is denoted by 0.

Example 1.3. The 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Note there are 2×3 zero matrices, 3×4 zero matrices, etc. In fact there is a zero matrix for every size.

Definition 1.4. (Equality of matrices) Let A and B be two matrices. Then $A = B$ means that the two matrices are of the same size and for $A = (a_{ij})$ and $B = (b_{ij})$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

The following properties of matrices can be easily verified. You should do so.

- Commutative law of addition.

$$A + B = B + A, \tag{1.1}$$

- Associative law for addition.

$$(A + B) + C = A + (B + C), \tag{1.2}$$

- Existence of an additive identity.

$$A + 0 = A, \tag{1.3}$$

- Existence of an additive inverse.

$$A + (-A) = 0. \tag{1.4}$$

Also for α, β scalars, the following additional properties hold.

- Distributive law over matrix addition.

$$\alpha (A + B) = \alpha A + \alpha B, \tag{1.5}$$

- Distributive law over scalar addition.

$$(\alpha + \beta) A = \alpha A + \beta A, \tag{1.6}$$

- Associative law for scalar multiplication.

$$\alpha (\beta A) = \alpha\beta (A), \tag{1.7}$$

- Rule for multiplication by 1.

$$1A = A. \tag{1.8}$$

As an example, consider the commutative law of addition. Let $A + B = C$ and $B + A = D$. Why is $D = C$?

$$C_{ij} = A_{ij} + B_{ij} = B_{ij} + A_{ij} = D_{ij}.$$

Therefore, $C = D$ because the ij^{th} entries are the same. Note that the conclusion follows from the commutative law of addition of numbers.

1.1.2 *Multiplication Of Matrices*

Definition 1.5. Matrices which are $n \times 1$ or $1 \times n$ are called **vectors** and are often denoted by a bold letter. Thus the $n \times 1$ matrix

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is also called a **column vector.** The $1 \times n$ matrix

$$(x_1 \cdots x_n)$$

is called a **row vector**.

Although the following description of matrix multiplication may seem strange, it is in fact the most important and useful of the matrix operations. To begin with, consider the case where a matrix is multiplied by a column vector. We will illustrate the general definition by first considering a special case.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = ?$$

This equals

$$7\begin{pmatrix} 1 \\ 4 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 5 \end{pmatrix} + 9\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}$$

In general, here is the definition of how to multiply an $(m \times n)$ matrix times a $(n \times 1)$ matrix.

Definition 1.6. Let $A = (A_{ij})$ be an $m \times n$ matrix

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$$

where the i^{th} column of A is denoted by $\mathbf{a}_i$, and let $\mathbf{v}$ be an $n \times 1$ matrix

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Then $A\mathbf{v}$ is an $m \times 1$ matrix equal to

$$v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n = \sum_{k=1}^{n} v_k\mathbf{a}_k$$

It follows from the observation that the j^{th} column of A is

$$\begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix},$$

that the above sum is of the form

$$v_1 \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{m1} \end{pmatrix} + v_2 \begin{pmatrix} A_{12} \\ A_{22} \\ \vdots \\ A_{m2} \end{pmatrix} + \cdots + v_k \begin{pmatrix} A_{1n} \\ A_{2n} \\ \vdots \\ A_{mn} \end{pmatrix}$$

It follows that the i^{th} entry of the $m \times 1$ matrix or column vector which results is

$$\sum_{j=1}^{n} A_{ij} v_j$$

Here is another example.

Example 1.4. Compute

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

First of all, this is of the form $(3 \times 4)(4 \times 1)$ and so the result should be a (3×1). Note how the inside numbers cancel. Then this equals

$$1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 5 \end{pmatrix}$$

The next task is to multiply an $m \times n$ matrix times an $n \times p$ matrix. Before doing so, the following may be helpful.

For A and B matrices, in order to form the product AB the number of columns of A must equal the number of rows of B.

$$(m \times \overbrace{n)\ (n}^{\text{these must match!}} \times p\ \ ) = m \times p$$

Note the two outside numbers give the size of the product. Remember:

The two middle numbers MUST match

Definition 1.7. When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix, B of the form

$$B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{pmatrix}$$

where $\mathbf{b}_k$ is an $n \times 1$ matrix or column vector. Then the $m \times p$ matrix AB is defined as follows:

$$AB \equiv \begin{pmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_p \end{pmatrix} \tag{1.9}$$

where $A\mathbf{b}_k$ is an $m \times 1$ matrix or column vector which gives the k^{th} column of AB.

Example 1.5. Multiply the following.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

The first thing you need to check before doing anything else is whether it is possible to do the multiplication. The first matrix is a 2×3 and the second matrix is a 3×3. Therefore, is it possible to multiply these matrices. According to the above discussion it should be a 2×3 matrix of the form

$$\left(\overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}^{\text{First column}}, \overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}}^{\text{Second column}}, \overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}^{\text{Third column}} \right)$$

You know how to multiply a matrix times a vector and so you do so to obtain each of the three columns. Thus

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{pmatrix}.$$

Example 1.6. Multiply the following.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

First check if it is possible. This is of the form $(3 \times 3)(2 \times 3)$. Aren't these the same two matrices considered in the previous example? Yes they are. It is just that here they are in a different order. This shows something you must always remember about matrix multiplication.

Order Matters!

This is very different than multiplication of numbers!

1.1.3 *The ij^{th} Entry Of A Product*

It is important to describe matrix multiplication in terms of entries of the matrices. What is the ij^{th} entry of AB? It would be the i^{th} entry of the j^{th} column of AB. Thus it would be the i^{th} entry of $A\mathbf{b}_j$. Now

$$\mathbf{b}_j = \begin{pmatrix} B_{1j} \\ \vdots \\ B_{nj} \end{pmatrix}$$