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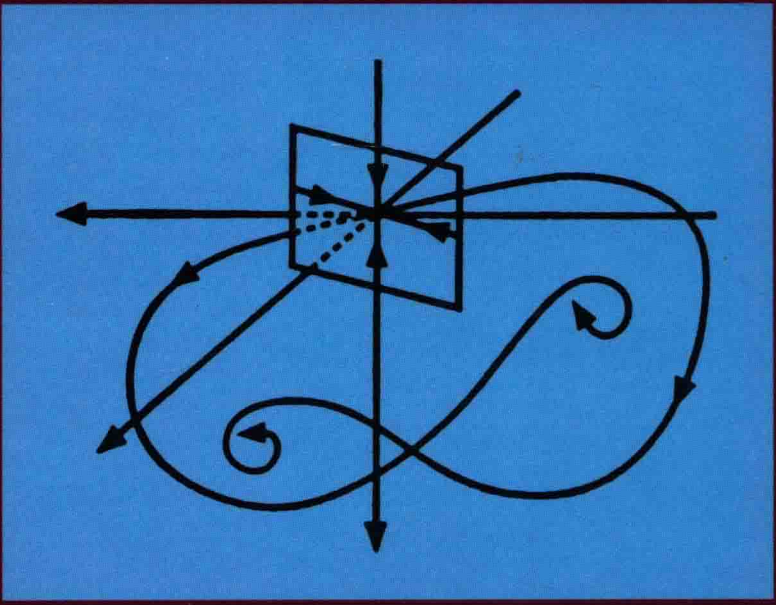
Dynamical Systems Method *for* Solving Operator Equations



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Dynamical Systems Method for Solving Operator Equations

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Preface

In this monograph a general method for solving operator equations, especially nonlinear and ill-posed, is developed. The method is called the dynamical systems method (DSM). Suppose one wants to solve an operator equation:

$$F(u) = 0, \tag{1}$$

where F is a nonlinear or linear map in a Hilbert or Banach space. We assume that equation (1) is solvable, possibly non-uniquely. The DSM for solving equation (1) consists of finding a map Φ such that the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0; \quad \dot{u} = \frac{du}{dt}, \tag{2}$$

has a unique global solution, i.e., solution $u(t)$ defined for all $t \geq 0$, there exists

$$u(\infty) = \lim_{t\to\infty} u(t), \text{ and } F(u(\infty)) = 0:$$

$$\exists! u \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = 0. \tag{3}$$

If (3) holds, we say that DSM is justified for equation (1). Thus the dynamical system in this book is a synonym to an evolution problem (2). This explains the name DSM. The choice of the initial data $u(0)$ will be discussed for various classes of equations (1). It turns out that for many classes of equations (1) the initial approximation u_0 can be chosen arbitrarily, and, nevertheless, (3) holds, while for some problems the choice of u_0, for which (3) can be established, is restricted to some neighborhood of a solution to equation (1).

We describe various choices of Φ in (2) for which it is possible to justify (3). It turns out that the scope of DSM is very wide. To describe it, let us introduce some notions. Let us call problem (1) *well-posed* if

$$\sup_{u\in B(u_0,R)} ||[F'(u)]^{-1}|| \leq m(R), \tag{4}$$

where $B(u_0, R) = \{u : ||u - u_0|| \le R\}$, $F'(u)$ is the Fréchet derivative (F-derivative) of the operator-function F at the point u, and the constant $m(R) > 0$ may grow arbitrarily as R grows. If (4) fails, we call problem (1) *ill-posed* . If problem (1) is ill-posed, we write it often as $F(u) = f$ and assume that noisy data f_δ are given in place of f, $||f_\delta - f|| \le \delta$. Although the equation $F(u) = f$ is solvable, the equation $F(u) = f_\delta$ may have no solutions.

The problem is:

Given $\{\delta, f_\delta, F\}$, find a stable approximation u_δ to a solution u of the equation $F(u) = f$, i.e., find u_δ such that

$$\lim_{\delta \to 0} ||u_\delta - u|| = 0. \tag{5}$$

Unless otherwise stated, we assume that

$$\sup_{u \in B(u_0, R)} ||F^{(j)}(u)|| \le M_j(R), \quad 0 \le j \le 2, \tag{6}$$

where $M_j(R)$ are some constants. In other words, we assume that the nonlinearity is C^2_{loc}, but the rate of its growth, as R grows, is not restricted.

Let us now describe briefly the scope of the DSM.

Any well-posed problem (1) can be solved by a DSM which converges at an exponential rate, i.e.,

$$||u(\infty) - u(t)|| \le re^{-c_1 t}, \quad ||F(u(t))|| \le ||F_0||e^{-c_1 t}, \tag{7}$$

where $r > 0$ and $c_1 > 0$ are some constants, and $F_0 := F(u_0)$.

For ill-posed problems, in general, it is not possible to estimate the rate of convergence; depending on the data f this rate can be arbitrarily slow. To estimate the rate of convergence in an ill-posed problem one has to make some additional assumptions about the data f. Remember that by "any" we mean throughout any solvable problem (1).

Any solvable linear equation

$$F(u) = Au - f = 0, \tag{8}$$

where A is a closed, linear, densely defined operator in a Hilbert space H, can be solved stably by a DSM. If noisy data f_δ are given, $||f_\delta - f|| \le \delta$, then DSM yields a stable solution u_δ for which (5) holds.

We derive stopping rules, i.e., rules for choosing $t(\delta) := t_\delta$, the time at which $u_\delta(t_\delta) = u_\delta$ should be calculated, using f_δ in place of f, in order for (5) to hold.

For linear problems (8) the convergence of a suitable DSM is global with respect to u_0, i.e., DSM converges to the unique minimal-norm solution of y of (8) for any choice of u_0.

Similar results we prove for equations (1) with monotone operators $F : H \to H$. Recall that F is called monotone if

$$(F(u) - F(v), u - v) \geq 0 \quad \forall u, v \in H, \tag{9}$$

where H is a Hilbert space. For hemicontinuous monotone operators the set $\mathcal{N} = \{u : \; F(u) = 0\}$ is closed and convex, and such sets in a Hilbert space have unique minimal-norm element. A map F is called hemicontinuous if the function $(F(u + \lambda v), w)$ is continuous with respect to $\lambda \in [0, \lambda_0)$ for any $u, v, w \in H$, where $\lambda_0 > 0$ is a number.

DSM is justified for any solvable equation (1) with monotone operators satisfying conditions (6). Note that no restrictions on the growth of $M_j(R)$ as R grows are imposed, so the nonlinearity is C^2_{loc} but may grow arbitrarily fast. For monotone operators we will drop assumption (6) and construct a convergent DSM.

We justify DSM for arbitrary solvable equation (1) in a Hilbert space with C^2_{loc} nonlinearity under a very weak assumption:

$$F'(y) \neq 0, \tag{10}$$

where y is a solution to equation (1).

We justify DSM for operators satisfying the following spectral assumption:

$$||(F'(u) + \varepsilon)^{-1}|| \leq \frac{c}{\varepsilon}, \quad 0 < \varepsilon \leq \varepsilon_0, \quad \forall u \in H, \tag{11}$$

where $\varepsilon_0 > 0$ is an arbitrary small fixed number. Assumption (11) is satisfied, for example, for operators $F'(u)$ whose regular points, i.e., points $z \in \mathbb{C}$ such that $(F'(u) - z)^{-1}$ is a bounded linear operator, fill in the set

$$|z| < \varepsilon_0, \quad |\arg z - \pi| \leq \varphi_0, \tag{12}$$

where $\varphi_0 > 0$ is an arbitrary small fixed number. We also prove the existence of a solution to the equation:

$$F(u) + \varepsilon u = 0, \tag{13}$$

provided that (6) and (11) hold.

We discuss DSM for equations (1) in Banach spaces. In particular, we discuss some singular perturbation problems for equations of the type (13): under what conditions a solution u_ε to equation (13) converges to a solution of equation (1) as $\varepsilon \to 0$.

In Newton-type methods, e.g.,

$$\dot{u} = -[F'(u)]^{-1} F(u), \quad u(0) = u_0, \tag{14}$$

the most difficult and time-consuming part is the inversion of the derivative $F'(u)$.

We propose a DSM method which avoids the inversion of the derivative.

For example, for well-posed problem (1) such a method is

$$\begin{aligned} \dot{u} &= -QF(u), \quad u(0) = u_0, \\ \dot{Q} &= -TQ + A^*, \quad Q(0) = Q_0, \end{aligned} \tag{15}$$

where

$$A := F'(u), \quad T = A^*A, \tag{16}$$

A^* is the adjoint to A operator, and u_0 and Q_0 are suitable initial approximations.

We also give a similar DSM scheme for solving ill-posed problem (1).

We justify DSM for some classes of operator equations (1) with unbounded operators, for example, for operators $F(u) = Au + g(u)$ where A is a linear, densely defined, closed operator in a Hilbert space H and g is a nonlinear C^2_{loc} map.

We justify DSM for equations (1) with some nonsmooth operators, e.g., with monotone, hemicontinuous, defined on all of H operators.

We show that the DSM can be used as a theoretical tool for proving conditions sufficient for the surjectivity of a nonlinear map or for this map to be a global homeomorphism.

One of our motivations is to develop a general method for solving operator equations, especially nonlinear and ill-posed. The other motivation is to develop a general approach to constructing convergent iterative processes for solving these equations.

The idea of this approach is straightforward: if the DSM is justified for solving equation (1), i.e., (3) holds, then one considers a discretization of (2), for example:

$$u_{n+1} = u_n + h_n \Phi(t_n, u_n), \quad u_0 = u_0, \quad t_{n+1} = t_n + h_n, \tag{17}$$

and if one can prove convergence of (17) to the solution of (2), then (17) is a convergent iterative process for solving equation (1).

We prove that any solvable linear equation (8) (with bounded or unbounded operator A) can be solved by a convergent iterative process which converges to the unique minimal-norm solution of (8) for any initial approximation u_0.

A similar result we prove for solvable equation (1) with monotone operators.

For general nonlinear equations (1), under suitable assumptions, a convergent iterative process is constructed. The initial approximation in this process does not have to be in a suitable neighborhood of a solution to (1).

We give some numerical examples of applications of the DSM. A detailed discussion of the problem of stable differentiation of noisy functions is given.

New technical tools, that we often usein this book, are some novel differential inequalities.

The first of these deals with the functions satisfying the following inequality:

$$\dot{g} \leq -\gamma(t)g(t) + \alpha(t)g^2(t) + \beta(t), \quad t \geq t_0 \geq 0, \tag{18}$$

where g, γ, α, β are nonnegative functions, and γ, α and β are continuous on $[t_0, \infty)$. We assume that there exists a positive function $\mu \in C^1[t_0, \infty)$, such that

$$0 \leq \alpha(t) \leq \frac{\mu(t)}{2}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \quad \beta(t) \leq \frac{1}{2\mu(t)}\left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right), \tag{19}$$

$$\mu(t_0)g(t_0) < 1, \tag{20}$$

and prove that under the above assumptions, any nonnegative solution $g(t)$ to (18) is defined on $[t_0, \infty)$ and satisfies the following inequality:

$$0 \leq g(t) \leq \frac{1 - \nu(t)}{\mu(t)} < \frac{1}{\mu(t)}, \tag{21}$$

where

$$\nu(t) = \frac{1}{\frac{1}{1-\mu(t_0)g(t_0)} + \frac{1}{2}\int_{t_0}^t \left(\gamma(s) - \frac{\dot{\mu}(s)}{\mu(s)}\right) ds}. \tag{22}$$

The other inequality, which we use, is an operator version of the Gronwall inequality. Namely, assume that:

$$\dot{Q} = -T(t)Q(t) + G(t), \quad Q(0) = Q_0, \tag{23}$$

where $T(t)$ and $G(t)$ are linear bounded operators on a Hilbert space depending continuously on a parameter $t \in [0, \infty)$. If there exists a continuous positive function $\varepsilon(t)$ on $[0, \infty)$ such that

$$(T(t)h, h) \geq \varepsilon(t)||h||^2 \quad \forall h \in H, \tag{24}$$

then the solution to (23) satisfies the inequality:

$$||Q(t)|| \leq e^{-\int_0^t \varepsilon(x)dx}\left[||Q_0|| + \int_0^t ||G(s)||e^{\int_0^s \varepsilon(x)dx} ds\right]. \tag{25}$$

This inequality shows that $Q(t)$ is a bounded linear operator whose norm is bounded uniformly with respect to t if

$$\sup_{t\geq 0}\int_0^t ||G(s)||e^{-\int_s^t \varepsilon(x)dx}ds < \infty. \tag{26}$$

The DSM is shown to be useful as a tool for proving theoretical results, see Chapter 13.

The DSM is used in Chapter 14 for construction of convergent iterative processes for solving operator equation.

In Chapter 15 some numerical problems are discussed, in particular, the problem of stable differentiation of noisy data.

In Chapter 16 various auxiliary material is presented. Together with some known results, available in the literature, some less known results are included: a necessary and sufficient condition for compactness of embedding operators and conditions for the continuity of the solutions to operator equations with respect to a parameter.

The table of contents gives a detailed list of topics discussed in this book.

Contents

Chapter 1

Introduction

1.1 What this book is about

This book is about a general method for solving operator equations

$$F(u) = 0. \tag{1.1.1}$$

Here F is a nonlinear map in a Hilbert space H. Later on we consider maps F in Banach spaces as well. The general method, that we develop in this book and call the dynamical systems method (DSM), consists of finding a nonlinear map $\Phi(t, u)$ such that the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0, \tag{1.1.2}$$

has a unique global solution $u(t)$, that is, the solution defined for all $t \geq 0$, this solution has a limit $u(\infty)$:

$$\lim_{t\to\infty} ||u(\infty) - u(t)|| = 0, \tag{1.1.3}$$

and this limit solves equation (1.1.1):

$$F(u(\infty)) = 0. \tag{1.1.4}$$

Let us write these three conditions as

$$\exists! u(t) \quad \forall t \geq 0; \quad \exists u(\infty); \quad F(u(\infty)) = 0. \tag{1.1.5}$$

If (1.1.5) holds for the solution to (1.1.2) then we say that a DSM is justified for solving equation (1.1.1). There may be many choices of $\Phi(t, u)$ for which DSM can be justified. A number of such choices will be given in

Chapter 3 and in other Chapters. It should be emphasized that we do not assume that equation (1.1.1) has a unique solution. Therefore the solution $u(\infty)$ depends on the initial approximation u_0 in (1.1.2). The choice of u_0 in some cases is not arbitrary and in many cases this choice is arbitrary, for example, for problems with linear operators, nonlinear monotone operators, and for a wide class of general nonlinear problems (see Chapters 4, 6, 7-9, 11-12, 14).

The existence and uniqueness of the local solution to problem (1.1.2) is guaranteed, for example, by a Lipschitz condition imposed on Φ:

$$||\Phi(t,u) - \Phi(t,v)|| \leq L||u-v||, \quad u,v \in B(u_0,R), \tag{1.1.6}$$

where the constant L does not depend on $t \in [0,\infty)$ and

$$B(u_0,R) = \{u: \quad ||u-u_0|| \leq R\}$$

is a ball, centered at the element $u_0 \in H$ and of radius $R > 0$.

1.2 What the DSM (Dynamical Systems Method) is

The DSM for solving equation (1.1.1) consists of finding a map $\Phi(t,u)$ and an initial element u_0 such that conditions (1.1.5) hold for the solution to the evolution problem (1.1.2).

If conditions (1.1.5) hold, then one solves Cauchy problem (1.1.2) and calculates the element $u(\infty)$. This element is a solution to equation (1.1.1). The important question one faces after finding a nonlinearity Φ, for which (1.1.5) hold, is the following one: how does one solve Cauchy problem (1.1.2) numerically? This question has been studied much in the literature. If one uses a projection method, i.e., looks for the solution of the form:

$$u(t) = \sum_{j=1}^{J} u_j(t) f_j, \tag{1.2.1}$$

where $\{f_j\}$ is an orthonormal basis of H, and $J > 1$ is an integer, then problem (1.1.2) reduces to a Cauchy problem for a system of J nonlinear ordinary differential equations for the scalar functions $u_j(t)$, $1 \leq j \leq J$, if the right-hand side of (1.1.2) is projected onto the J-dimensional subspace spanned by $\{f_j\}_{1 \leq j \leq J}$. This system is:

$$\dot{u}_j = \left(\Phi(\sum_{m=1}^{J} u_m(t) f_m, t), f_j \right), \quad 1 \leq j \leq J, \tag{1.2.2}$$

$$u_j(0) = (u_0, f_j), \quad 1 \leq j \leq J. \tag{1.2.3}$$

Numerical solution of the Cauchy problem for systems of ordinary differential equations has been much studied in the literature.

In this book the main emphasis is on the possible choices of Φ which imply properties (1.1.5).

1.3 The scope of the DSM

One of our aims is to show that DSM is applicable to a very wide variety of problems.

Specifically, we prove in this book that the DSM is applicable to the following classes of problems:

1. *Any well-posed solvable problem (1.1.1) can be solved by DSM.*

   By a *well-posed problem* (1.1.1) we mean the problem with the operator F satisfying the following assumptions:

   $$\sup_{u\in B(u_0,R)} ||[F'(u)]^{-1}|| \le m(R), \tag{1.3.1}$$

   and

   $$\sup_{u\in B(u_0,R)} ||F^{(j)}(u)|| \le M_j(R), \quad 0 \le j \le 2, \tag{1.3.2}$$

   where $F^{(j)}(u)$ is the j-th Fréchet derivative of F.

   If assumption (1.3.1) does not hold, but (1.3.2) holds, we call problem (1.1.1) *ill-posed*. This terminology is not quite standard. The standard notion of an ill-posed problem is given in Section 2.1.

   We prove that for any solvable well-posed problem not only the DSM can be justified, i.e., Φ can be found such that for problem (1.1.2) conclusions (1.1.5) hold, but, in addition, the convergence of $u(t)$ to $u(\infty)$ is exponentially fast:

   $$||u(t) - u(\infty)|| \le re^{-c_1 t}, \tag{1.3.3}$$

   where $r > 0$ and $c_1 > 0$ are constants, and

   $$||F(u(t))|| \le ||F_0||e^{-c_1 t}, \quad F_0 := F(u_0). \tag{1.3.4}$$

2. *Any solvable linear ill-posed problem can be solved by DSM.*

   A linear problem (1.1.1) is a problem

   $$Au = f, \tag{1.3.5}$$