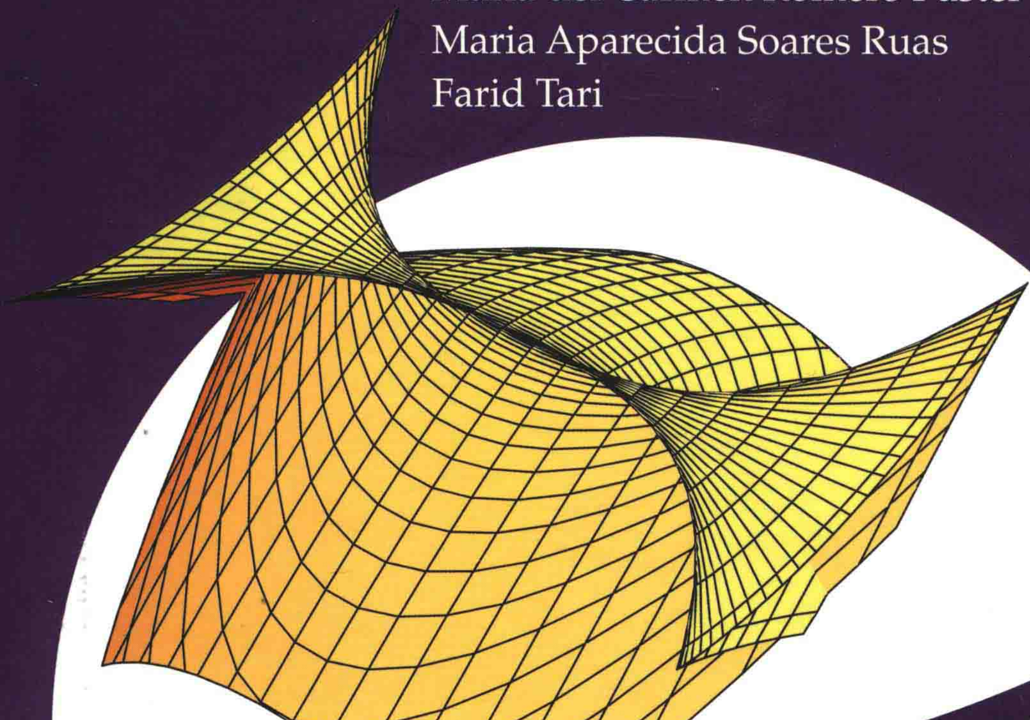
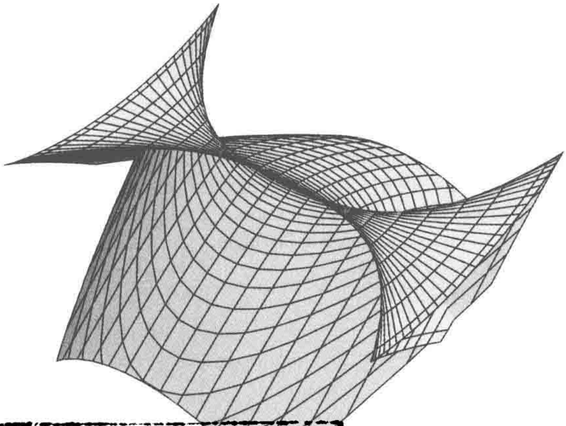
Differential Geometry from a Singularity Theory Viewpoint

Shyuichi Izumiya
Maria del Carmen Romero Fuster
Maria Aparecida Soares Ruas
Farid Tari



World Scientific

Differential Geometry from a Singularity Theory Viewpoint



Shyuichi Izumiya
Hokkaido University, Japan

Maria del Carmen Romero Fuster
Universitat de València, Spain

Maria Aparecida Soares Ruas
University of São Paulo, Brazil

Farid Tari
University of São Paulo, Brazil

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Differential geometry from a singularity theory viewpoint / by Shyuichi Izumiya (Hokkaido University, Japan) [and three others].
pages cm
Includes bibliographical references and index.
ISBN 978-9814590440 (hardcover : alk. paper)
1. Surfaces--Areas and volumes. 2. Singularities (Mathematics) 3. Geometry, Differential. 4. Curvature. I. Izumiya, Shyuichi.
QA645.D54 2015
516.3'6--dc23

2015033184

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore by B & Jo Enterprise Pte Ltd

Differential Geometry from a Singularity Theory Viewpoint

Preface

The geometry of surfaces is a subject that has fascinated many mathematicians and users of mathematics. This book offers a new look at this classical subject, namely from the point of view of singularity theory. Robust geometric features on a surface in the Euclidean 3-space, some of which are detectable by the naked eye, can be captured by certain types of singularities of some functions and mappings on the surface. In fact, the mappings in question come as members of some natural families of mappings on the surface. The singularities of the individual members of these families of mappings measure the contact of the surface with model objects such as lines, circles, planes and spheres.

This book gives a detailed account of the theory of contact between manifolds and its link with the theory of caustics and wavefronts. It then uses the powerful techniques of these theories to deduce geometric information about surfaces immersed in the Euclidean 3, 4 and 5-spaces as well as spacelike surfaces in the Minkowski space-time.

In Chapter 1 we argue the case for using singularity theory to study the extrinsic geometry of submanifolds of Euclidean spaces (or of other spaces). To make the book self-contained, we devote Chapter 2 to introducing basic facts about the extrinsic geometry of submanifolds of Euclidean spaces. Chapter 3 deals with singularities of smooth mappings. We state the results on finite determinacy and versal unfoldings which are fundamental in the study of the geometric families of mappings on surfaces treated in the book. Chapter 4 is about the theory of contact introduced by Mather and developed by Montaldi. In Chapter 5 we recall some basic concepts in symplectic and contact geometry and establish the link between the theory of contact and that of Lagrangian and Legendrian singularities. We apply in Chapters 6, 7 and 8 the singularity theory framework exposed in the previous chapters to the study of the extrinsic differential geometry of surfaces in the Euclidean 3, 4 and 5-spaces respectively. The codimension of the surface in the ambient space is 1, 2 or 3 and this book shows how some aspects of the geometry of the surface change with its codimension. In Chapter 9 we chose spacelike surfaces in the Minkowski space-time to illustrate how to approach the study of submanifolds in Minkowski spaces using singularity theory. Most of the results in the previous chapters are local in nature. Chapter 10 gives a flavour of global results on closed surfaces using local invariants obtained from the local study of the surfaces in the previous chapters.

The emphasis in this book is on how to apply singularity theory to the study of the extrinsic geometry of surfaces. The methods apply to any smooth submanifolds of higher dimensional Euclidean space as well as to other settings, such as affine, hyperbolic or Minkowski spaces. However, as it is shown in Chapters 6, 7 and 8, each pair (m, n) with m the dimension of the submanifold and n of the ambient space needs to be considered separately.

This book is unapologetically biased as it focuses on research results and interests of the authors and their collaborators. We tried to remedy this by including, in the Notes of each chapter, other aspect and studies on the topics in question and as many references as we can. Omissions are inevitable, and we apologise to anyone whose work is unintentionally left out.

Currently, there is a growing and justified interest in the study of the differential geometry of singular submanifolds (such as caustics, wavefronts, images of singular mappings etc) of Euclidean or Minkowski spaces, and of submanifolds with induced (pseudo) metrics changing signature on some subsets of the submanifolds. We hope that this book can be used as a guide to anyone embarking on the study of such objects.

This book has been used (twice so far!) by the last-named author as lecture notes for a post-graduate course at the University of São Paulo, in São Carlos. We thank the following students for their thorough reading of the final draft of the book: Alex Paulo Francisco, Leandro Nery de Oliveira, Lito Edinson Bocanegra Rodríguez, Martin Barajas Sichaca, Mostafa Salarinoghabi and Patricia Tempesta. Thanks are also due to Catarina Mendes de Jesus for her help with a couple of the book's figures and to Asahi Tsuchida, Shunichi Honda and Yutaro Kabata for correcting some typos. Most of the results in Chapter 4 are due to James Montaldi. We thank him for allowing us to reproduce some of his proofs in this book.

We are also very grateful to Masatomo Takahashi for reading the final draft of the book and for his invaluable comments and corrections.

S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari
August, 2015

Contents

Chapter 1

The case for the singularity theory approach

The study of curves and surfaces in the Euclidean space is a fascinating and important subject in differential geometry. We highlight in this chapter how singularity theory can be used not only to recover classical results on curves and surfaces in a simpler and more elegant way but also how it reveals the rich and deep underlying concepts involved.

We start with the evolute and parallels of a plane curve. We first use classical differential geometry techniques to obtain the shape of the evolute and parallels. We then define the family of distance squared functions on the plane curve and recover from the singularities type of the members of this family geometric information about the curve itself. We outline how to use the Lagrangian and Legendrian singularity theory framework to deduce properties of the evolute that are invariant under diffeomorphisms. We proceed similarly for surfaces in the Euclidean 3-space and consider the singularities of their focal sets. We deal in the last section with the singularities of ruled and developable surfaces.

We refer to [do Carmo (1976)] for a detailed study of the differential geometry of curves and surfaces.

Throughout this book, a given map is said to be *smooth* (or C^∞) if its partial derivatives of all order exist and are continuous.

The Euclidean n-space is the vector space $\mathbb{R}^n$ endowed with the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$

for any $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$.

We also view the Euclidean n-space as a set of points. The vector space $\mathbb{R}^n$ comes with a standard orthogonal basis $\mathbf{e}_1 = (1, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, \ldots, 1)$. We choose a point $O = (0, \ldots, 0)$ to be the origin and denote by $\Sigma = (O, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ the standard orthonormal coordinates system in $\mathbb{R}^n$. Then, a point p in the Euclidean n-space is the endpoint of the vector Op and its coordinates $(x_1, \dots, x_n)$ in the system Σ are the coordinates of the vector Op in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Curves, surfaces, submanifolds in $\mathbb{R}^n$ are considered as subsets of points in $\mathbb{R}^n$.

The vector product of $n-1$ vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ in $\mathbb{R}^n$, is defined by

$$\mathbf{u}_1 \times \cdots \times \mathbf{u}_{n-1} = \begin{vmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ u_1^1 & \cdots & u_n^1 \\ \vdots & \cdots & \vdots \\ u_1^{n-1} & \cdots & u_n^{n-1} \end{vmatrix},$$

where $\mathbf{u}_i = (u_1^i, \dots, u_n^i)$. By the property of the determinant, we have

$$\langle \mathbf{u}, \mathbf{u}_1 \times \cdots \times \mathbf{u}_{n-1} \rangle = \det(\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}).$$

1.1 Plane curves

A smooth curve in the Euclidean n-space is a smooth map $\gamma : I \to \mathbb{R}^n$, where I is an open interval of $\mathbb{R}$. The *trace* of γ, which we still denote by γ, is the set of points $\gamma(I)$ in $\mathbb{R}^n$. The curve γ is said to be *regular* if $\gamma'(t)$ is not the zero vector for any t in I. Points where $\gamma'(t)$ is the zero vector are called *singular points* of γ.

We consider here smooth and regular plane curves ($n = 2$ above). We shall suppose that the curve $\gamma : I \to \mathbb{R}^2$ is parametrised by arc length and denote the arc length parameter by s. Then, $\mathbf{t}(s) = \gamma'(s)$ is a unit tangent vector to γ. We denote by $\mathbf{n}(s)$ the unit normal vector to γ obtained by rotating $\mathbf{t}(s)$ anti-clockwise by an angle of $\pi/2$. It follows from the fact that $\langle \mathbf{t}(s), \mathbf{t}(s) \rangle = 1$ that $\langle \mathbf{t}'(s), \mathbf{t}(s) \rangle = 0$, so

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \tag{1.1}$$

for some smooth function $\kappa(s)$, called *the curvature* of γ at s.

We have, similarly, $\langle \mathbf{n}'(s), \mathbf{n}(s) \rangle = 0$, so $\mathbf{n}'(s) = \alpha(s)\mathbf{t}(s)$ for some function $\alpha(s)$. Differentiating the identity $\langle \mathbf{t}(s), \mathbf{n}(s) \rangle = 0$ and using (1.1) gives $\alpha(s) = -\kappa(s)$, so that

$$\mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s).$$

We can use (1.1) to deduce that

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle.$$