ADVANCED CALCULUS

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TO MY WIFE
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Preface

This book has grown out of my experience in teaching advanced calculus over a period of more than a dozen years. It has taken shape gradually as I observed the effect on my students of various presentations of theory and technique, and as I accumulated experience in the construction of problems and examination questions. Every part of the book has been planned to make the whole an effective instrument for imparting the fundamental principles and methods of analysis to students at the advanced calculus level. The book is aimed at the student reader; I have striven to arouse interest at every stage, to motivate the direction of the exposition, and to achieve clarity through ample illustrative examples and particular care in directing the course of the reasoning.

In many of the chapters the first section is devoted to one or more of the following: (1) setting forth in general terms the aim of the chapter, (2) supplying motivation for the subject matter, (3) explaining my point of view in fitting the chapter into the book as a whole.

Books on advanced calculus vary widely in choice of subject matter, in emphasis on particular topics, and in treatment of the relation between elementary and advanced calculus. These are matters on which no one book can fully please all users. For lack of space I have omitted treatment of some topics which I would have liked to include. The emphasis is on sound understanding of concepts, and on the basic principles of analysis: those properties of the real number system which support the theory of limits and continuity. But the thread of theoretical development is imbedded in an ample exposition of the methods and techniques which are needed by every student of advanced applied mathematics There is a generous and rich supply of exercises and problems.

Learning in calculus is cumulative. It is also evolutionary. The student does not come all at once to a one and only correct understanding of new ideas. At each new level of his maturity he can gain a fresh appreciation of things he has already been taught. An advanced calculus should not ignore or discard all that a student already knows about calculus. Rather, it should build upon what he knows, and strengthen that knowledge by emphasis upon those aspects of elementary calculus which are given least attention in the usual freshman and sophomore courses, and which become increasingly important as a student progresses into more advanced analysis. Chapter I of the present book is designed to be used for building in this way.

The book is written on the assumption that students using it have normal skill in the formal aspects of elementary calculus, and that they can draw freely on the standard formulas of algebra, trigonometry, and calculus relative to the elementary functions. Some of the logical issues pertaining to the definition of logarithms, exponentials, and trigonometric functions are not fully met in elementary calculus, of course. But I prefer not to tackle these issues prematurely. They can be settled in due time, and in a variety of ways, once the student knows enough about definite integrals, infinite series, and uniform convergence. Meanwhile, the student is eager to get on to new ideas, new techniques, new applications.

A word about the system of numbering. Sections within each chapter are numbered in decimal order. Thus, in Chapter XVII, § 17.21 and § 17.22 follow each other between § 17.2 and § 17.3. The first section in each chapter has no digits after the decimal point. Formulas in each section are numbered with consecutive integers after a dash which follows the section number in which the formulas occur. Thus, formulas in § 12.31 range from (12.31–1) to (12.31–6). Theorems are numbered consecutively in Roman numerals, starting with Theorem I for the first theorem in each new chapter. References to theorems usually cite the theorem number and the section in which it occurs.

I have not attempted to include a bibliography. I have been influenced by many books, both American and European, but I cannot account for the influences in detail.

In sending the book forth I pay my respects to the memory of one of my teachers, Professor William Fogg Osgood, and I thank heartily all those students and colleagues who have taken an interest in seeing the book brought to completion.

ANGUS E. TAYLOR

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ADVANCED CALCULUS

CHAPTER I

Fundamentals of Elementary Calculus

1. Introduction

A course in advanced calculus must build upon the presumption that students studying the subject have already gained some knowledge of elementary calculus. We shall therefore begin by taking a backward look over those parts of calculus with which the reader of this book should have facility and a measure of understanding. Our object in such a retrospect is not to conduct a systematic review. The purpose is, rather, to establish a common point of view for students whose training in calculus, up to this point, must inevitably reflect a wide variety of practices in teaching, choice of subject matter, and distribution of emphasis between the acquisitions of problem-solving skills and mastery of fundamental theory. As we survey the field of elementary calculus we shall stress the conceptual aspect of the subject: fundamental definitions and processes which underlie all the applications. In a first course in calculus it is often the case that the fundamental notions are introduced through the medium of particular geometrical or physical applications. Thus, to the beginner, the derivative may be typified by, or even identified with, the speed of a moving object, while the integral is thought of as the area under a curve. We now seek to take a more general, or abstract, view. Differentiation and integration are processes which are carried out upon functions. We need to have a clear understanding of the definitions of these processes, quite apart from their applications.

Another aspect of our survey will be our concern with the logical unfolding of the fundamental principles of calculus. Here again we strive to take a more mature point of view. We wish to indicate in what respects it is desirable and necessary to look more deeply into the derivations of rules and proofs of theorems. There are places in elementary calculus, as usually taught to beginners, where the development is necessarily inadequate from the standpoint of logic. In many places the reasoning leans heavily on intuition or on one sort or another of plausibility argument. That this state of affairs persists is partly due to a deliberate placing of emphasis: we make our primary goal the attainment of skill in the manipulative techniques of calculus which lend themselves readily to applications at an elementary level in physics, engineering, and the like. This kind of skill (up to a certain point) can be imparted without paying much attention to questions of logical rigor. But it is also true that there are logical inadequacies in a first course in calculus which cannot be made good entirely within the customary time limits of such a course (two or three semesters), even where a reasonably heavy emphasis is laid upon "theory." At bottom the subject of calculus rests upon the real number system and the theory of limits. A full appreciation and understanding of this foundation material must come slowly, but the need for such understanding becomes more acute as we progress in learning. In advanced calculus we must make a deeper study of the real number system, of the theory of limits, and of the properties of continuous functions. In this way only can we proceed easily and with confidence to a mastery of many new concepts and processes of higher mathematics.

1.1 Functions

At the very outset we must discuss the mathematical concept of a *function*, for we shall constantly be talking about properties of functions and about processes which are applied to functions. The function concept has been very much generalized since the early development of calculus by Leibniz and Newton. At the present time the word "function" is used broadly to mean any determinate correspondence between two classes of objects.

EXAMPLE 1. Consider the class of all plane polygons. If to each polygon we make correspond the number which is the perimeter of the polygon (in terms of some fixed unit of length), this correspondence is a function. Here the first class of objects is composed of certain geometrical figures, while the members of the second class are positive numbers.

To begin with, let us consider functions which are correspondences between classes of real numbers. Such functions are called *real functions of a real variable.* The first class of numbers is the *domain of definition* of the function. Once this domain (call it D) has been specified, the function is defined as soon as a definite rule of correspondence has been given, assigning to each number of D some corresponding number (or numbers). If x is a symbol which may be used to denote any member of D, we call x the *independent variable* of the function. Sometimes there may be more than one number corresponding to a given value of x; in this case the function is said to be *multiple-valued.* If there is just one number corresponding to each value of x, the function is said to be *single-valued.* We usually find it possible to deal with multiple-valued functions by sepa-