Nankai Tracts in Mathematics Vol. 14

ETALE COHOMOLOGY THEORY

Revised Edition

Lei Fu

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NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI · CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Fu, Lei.
Etale cohomology theory / Lei Fu (Chern Institute of Mathematics, Nankai University, China). -- Revised edition.
pages cm. -- (Nankai tracts in mathematics ; volume 14)
Includes bibliographical references and index.
ISBN 978-9814675086 (hardcover : alk. paper)
1. Homology theory. 2. Arithmetical algebraic geometry. I. Title.
QA612.3.F8 2015
514'.23--dc23
2015002795

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore by B & Jo Enterprise Pte Ltd

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NANKAI TRACTS IN MATHEMATICS

Series Editors: Yiming Long, Weiping Zhang and Lei Fu
Chern Institute of Mathematics

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Preface

The manuscript of this book was written in 1998–1999. At that time, I was invited to give a series of talks at the Morningside Center of Mathematics on Deligne's proof of the Weil conjecture. To prepare the talks, I wrote the book [Fu (2006)] on algebraic geometry covering the main materials in [EGA] I–III, and the current book covering the main materials in [SGA] 1, 4, $4\frac{1}{2}$, and 5 related to etale cohomology theory. I hope this book provides adequate preparation for reading more advanced papers such as [Beilinson, Bernstein and Deligne (1982)], [Deligne (1974)], [Deligne (1980)] and [Laumon (1987)].

The prerequisites for reading this book are [Fu (2006)] and the book [Matsumura (1970)] on commutative algebra. As [Fu (2006)] may not be widely available, whenever a result from it is quoted, a corresponding result in [EGA] or [Hartshorne (1977)] is also indicated. A result used in this book but not covered in these books is Artin's approximation theorem [Artin (1969)]. A nice account can be found in [Bosch, Lütkebohmert and Raynaud (1990)].

At the beginning of each section, I give a list of references related to the content of this section. I strongly encourage the reader to go through these references, especially [SGA], for more general and thorough treatment. When I was a graduate student, the books [Freitag and Kiehl (1988)] and [Milne (1980)] on etale cohomology theory gave me great help for reading [SGA]. It is inevitable that some treatments in this book are influenced by them.

I would like to thank Jiangxue Fang, Enlin Yang, Takeshi Saito and Hao Zhang for pointing out errors, misprints, and improvement of an earlier edition of this book. During the preparation of the book, I am supported by the Qiu Shi Science & Technologies Foundation and the NSFC.

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Contents

Chapter 1

Descent Theory

Unless otherwise stated, rings in this book are commutative with the identity element 1, and homomorphisms of rings map 1 to 1. For any ring A and any A-module M, we assume $1 \cdot x = x$ for all $x \in M$. For any scheme $(S, \mathscr{O}_S)$ and any $s \in S$, denote the maximal ideal of $\mathscr{O}_{S,s}$ by $\mathfrak{m}_s$, and denote the residue field of $\mathscr{O}_{S,s}$ by $k(s)$. For any nonnegative integer n, denote by $\mathbb{A}^n_S$ the affine space $\mathbf{Spec}\,\mathscr{O}_S[t_1, \ldots, t_n]$ over S, and by $\mathbb{P}^n_S$ the projective space $\mathbf{Proj}\,\mathscr{O}_S[t_0, t_1, \ldots, t_n]$ over S. We identify $\mathbb{A}^n_S$ with the open subscheme $\mathbf{Spec}\,\mathscr{O}_S[\frac{t_1}{t_0}, \ldots, \frac{t_n}{t_0}]$ of $\mathbb{P}^n_S$.

1.1 Flat Modules

([SGA 1] IV 1.)

Let A be a ring. An A-module M is called *flat* if the functor $N \mapsto M \otimes_A N$ on the category of A-modules is exact. We also say that M is flat over A, or A-flat. Let $A \to B$ be a homomorphism of rings. If M is a flat A-module, then $B \otimes_A M$ is a flat B-module. If N is a flat B-module, and B is flat over A, then N is flat over A.

Proposition 1.1.1. *Let A be a ring and M an A-module. The following conditions are equivalent:*

(i) *M is flat.*

(ii) *For any A-modules N, we have $\mathrm{Tor}_i^A(M, N) = 0$ for all $i \geq 1$.*

(iii) *For any finitely generated A-module N, we have $\mathrm{Tor}_i^A(M, N) = 0$ for all $i \geq 1$.*

(iv) *For any A-module N, we have $\mathrm{Tor}_1^A(M, N) = 0$.*

(v) *For any finitely generated A-module N, we have $\mathrm{Tor}_1^A(M, N) = 0$.*

(vi) *For any ideal I of A, we have $\mathrm{Tor}_1^A(M, A/I) = 0$.*

(vii) *For any finitely generated ideal I of A, we have* $\mathrm{Tor}_1^A(M, A/I) = 0$.

(viii) *For any ideal I of A, the canonical homomorphism*

$$I \otimes_A M \to M, \quad a \otimes x \mapsto ax$$

is injective, that is, it induces an isomorphism $I \otimes_A M \cong IM$.

(ix) *For any finitely generated ideal I of A, the canonical homomorphism*

$$I \otimes_A M \to M, \quad a \otimes x \mapsto ax$$

is injective.

Let M be a flat A-module, N an A-module, N' and N'' submodules of N. Then $M \otimes_A N'$ and $M \otimes_A N''$ can be regarded as submodules of $M \otimes_A N$. We have

$$M \otimes_A (N' \cap N'') \cong (M \otimes_A N') \cap (M \otimes_A N''),$$
$$M \otimes_A (N' + N'') \cong (M \otimes_A N') + (M \otimes_A N''),$$

where on the right-hand side, we take the intersection and the summation inside $M \otimes_A N$.

Proposition 1.1.2.

(i) *Let A be a ring and let S be a multiplicative subset in A. Then $S^{-1}A$ is flat over A. If M is a flat A-module, then $S^{-1}M$ is a flat $S^{-1}A$-module.*

(ii) *Let $A \to B$ be a homomorphism of rings, let S (resp. T) be a multiplicative subset in A (resp. B) such that the image of S in B is contained in T, and let N be a B-module. If N is flat over A, then $T^{-1}N$ is flat over A and over $S^{-1}A$.*

(iii) *Let $A \to B$ be a homomorphism of rings and let N be a B-module. Suppose for every maximal ideal $\mathfrak{n}$ of B, $N_\mathfrak{n}$ is flat over A. Then N is flat over A.*

Proof. Let us prove (ii). For any A-module M, we have

$$T^{-1}N \otimes_A M \cong T^{-1}(N \otimes_A M).$$

If N is flat over A, the functor $T^{-1}(N \otimes_A -)$ on the category of A-modules is exact. It follows that $T^{-1}N$ is flat over A. By (i), $S^{-1}T^{-1}N$ is flat over $S^{-1}A$. We have $S^{-1}T^{-1}N \cong T^{-1}N$. □

Proposition 1.1.3.

(i) *Let A be a ring and let M be a flat A-module. If $a \in A$ is not a zero divisor, then the canonical homomorphism*

$$M \to M, \quad x \mapsto ax$$

is injective. In particular, if A is an integral domain, then M has no torsion.

(ii) *Let A be an integral domain such that $A_{\mathfrak{m}}$ is a discrete valuation ring for every maximal ideal $\mathfrak{m}$ of A. Then an A-module M is flat if and only if it has no torsion.*

Proof. Let us prove the "if" part of (ii). Suppose M has no torsion. To prove M is A-flat, it suffices to show $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat for any maximal ideal $\mathfrak{m}$ of A. Let I be an ideal of $A_{\mathfrak{m}}$. By our assumption, I is principal, say generated by some element $r \in A$. The canonical map

$$A_{\mathfrak{m}} \to I, \quad a \mapsto ra$$

is an isomorphism. So we have an isomorphism

$$M_{\mathfrak{m}} \stackrel{\cong}{\to} I \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}, \quad x \mapsto r \otimes x.$$

The composite of this isomorphism with the canonical homomorphism

$$I \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \to M_{\mathfrak{m}}, \quad a \otimes x \to ax$$

is

$$M_{\mathfrak{m}} \to M_{\mathfrak{m}}, \quad x \mapsto rx,$$

which is injective since M has no torsion. We then apply 1.1.1 (viii). □

1.2 Faithfully Flat Modules

([SGA 1] IV 2–4.)

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. We say that F is *faithful* if for all objects X and Y in $\mathscr{C}$, the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)), \quad f \mapsto F(f)$$

is injective. If $\mathscr{C}$ and $\mathscr{D}$ are additive categories and F is an additive functor, then the above condition is equivalent to saying that the condition $F(u) = 0$ implies the condition $u = 0$ for any $u \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. In this case, the condition $F(X) = 0$ implies the condition $X = 0$ for any object X in $\mathscr{C}$. Indeed, we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} = 0$, and hence $\mathrm{id}_X = 0$.

A functor $F : \mathscr{C} \to \mathscr{D}$ is called *fully faithful* if the map

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)), \quad f \mapsto F(f)$$

is bijective for all objects $X, Y \in \mathrm{ob}\,\mathscr{C}$. F is called *essentially surjective* if for any object Z in $\mathscr{D}$, there exists an object X in $\mathscr{C}$ such that $F(X) \cong Z$.

We say that F is an *equivalence of categories* if F is fully faithful and essentially surjective.

Proposition 1.2.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories and let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor. The following conditions are equivalent:*

(i) *F is exact and faithful.*

(ii) *A sequence*

$$M' \to M \to M''$$

in $\mathscr{C}$ is exact if and only if

$$F(M') \to F(M) \to F(M'')$$

is exact.

(iii) *F is exact and the condition $F(X) = 0$ implies the condition $X = 0$.*

Suppose furthermore that there exists a family of nonzero objects $\{Z_i\}$ in $\mathscr{C}$ such that for any nonzero object X in $\mathscr{C}$, there exist some Z_i and some object Y in $\mathscr{C}$ admitting a monomorphism $Y \to X$ and an epimorphism $Y \to Z_i$. Then the above conditions are equivalent to the following:

(iv) *F is exact and $F(Z_i) \neq 0$ for all Z_i.*

Proof.
(i)⇒(ii) Given a sequence

$$M' \stackrel{u}{\to} M \stackrel{v}{\to} M''$$

in $\mathscr{C}$, suppose

$$F(M') \stackrel{F(u)}{\to} F(M) \stackrel{F(v)}{\to} F(M'')$$

is exact. We have $F(vu) = F(v)F(u) = 0$. Since F is faithful, we have $vu = 0$. Hence $\operatorname{im} u \subset \ker v$. Since F is exact, we have

$$F(\ker v/\operatorname{im} u) \cong F(\ker v)/F(\operatorname{im} u) \cong \ker F(v)/\operatorname{im} F(u) = 0.$$

Hence $\ker v/\operatorname{im} u = 0$, that is, $\ker v = \operatorname{im} u$.

(ii)⇒(iii) If $F(X) = 0$, then

$$F(0) \to F(X) \to F(0)$$

is exact. Our condition implies that

$$0 \to X \to 0$$

is exact. So $X = 0$.

(iii)⇒(i) Let $u : X \to Y$ be a morphism in $\mathscr{C}$. If $F(u) = 0$, then $\operatorname{im} F(u) = 0$. Since F is exact, we have $F(\operatorname{im} u) \cong \operatorname{im} F(u) = 0$. By our condition, we have $\operatorname{im} u = 0$, that is, $u = 0$.

(iii)$\Rightarrow$ (iv) is obvious.

(iv)$\Rightarrow$(iii) For any nonzero object X in $\mathscr{C}$, choose an object Y admitting a monomorphism $Y \to X$ and an epimorphism $Y \to Z_i$. Then $F(Y) \to F(Z_i)$ is an epimorphism. As $F(Z_i) \neq 0$, we have $F(Y) \neq 0$. The morphism $F(Y) \to F(X)$ is a monomorphism. It follows that $F(X) \neq 0$. □

Corollary 1.2.2. *Let A be a ring and M an A-module. The following conditions are equivalent:*

(i) *The functor $N \mapsto M \otimes_A N$ on the category of A-modules is exact and faithful.*

(ii) *A sequence of A-modules*

$$N' \to N \to N''$$

is exact if and only if

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$$

is exact.

(iii) *M is flat and the condition $M \otimes_A N = 0$ implies the condition $N = 0$.*

(iv) *M is flat and $M \otimes_A A/\mathfrak{m} \neq 0$ for any maximal ideal $\mathfrak{m}$ of A.*

When M satisfies the above equivalent conditions, we say that M is *faithfully flat.*

Corollary 1.2.3. *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of local rings and let M be a finitely generated B-module. Then M is faithfully flat over A if and only if it is flat over A and nonzero.*

Indeed, by Nakayama's lemma, the condition $M \otimes_A A/\mathfrak{m} \neq 0$ is equivalent to the condition $M \neq 0$.

Proposition 1.2.4. *Let $A \to B$ be a homomorphism of rings. If there exists a B-module M faithfully flat over A, then the map* $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ *is onto.*

Proof. It suffices to show that for any $\mathfrak{p} \in \mathrm{Spec}\, A$, the fiber $\mathrm{Spec}\,(B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$ of the map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ over $\mathfrak{p}$ is not empty, or equivalently, $B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is nonzero. Indeed, since M is faithfully flat over A, $M \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is faithfully flat over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. This implies that $M \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \neq 0$. But $M \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is a $(B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$-module. So $B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \neq 0$. □

Corollary 1.2.5. *Let $\phi : A \to B$ be a homomorphism of rings and let M be a finitely generated B-module flat over A. Suppose* $\operatorname{Supp} M = \operatorname{Spec} B$*. Then for any $\mathfrak{p} \in \operatorname{Spec} A$, and any prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ which is minimal among those prime ideals of B containing $\mathfrak{p}B$, we have $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. In particular, for any minimal prime ideal $\mathfrak{q}$ of B, $\phi^{-1}(\mathfrak{q})$ is a minimal prime ideal of A.*

Proof. By 1.2.3, $M_{\mathfrak{q}}$ is faithfully flat over $A_{\phi^{-1}(\mathfrak{q})}$. By 1.2.4, the map $\operatorname{Spec} B_{\mathfrak{q}} \to \operatorname{Spec} A_{\phi^{-1}(\mathfrak{q})}$ is onto. We have $\mathfrak{p}A_{\phi^{-1}(\mathfrak{q})} \in \operatorname{Spec} A_{\phi^{-1}(\mathfrak{q})}$. By the minimality of $\mathfrak{q}$, the preimage of $\mathfrak{p}A_{\phi^{-1}(\mathfrak{q})}$ in $\operatorname{Spec} B_{\mathfrak{q}}$ must be $\mathfrak{q}B_{\mathfrak{q}}$. It follows that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. □

Proposition 1.2.6. *Let $\phi : A \to B$ be a homomorphism of rings. The following conditions are equivalent:*

(i) *B is faithfully flat over A.*

(ii) *B is flat over A and* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is onto.*

(iii) *B is flat over A, and for every maximal ideal $\mathfrak{m}$ of A, there exists a maximal ideal $\mathfrak{n}$ of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.*

(iv) *B is flat over A, and for any A-module M, the canonical homomorphism*

$$M \to M \otimes_A B, \quad x \mapsto x \otimes 1$$

is injective.

(v) *For every ideal I of A, the canonical homomorphism*

$$I \otimes_A B \to B, \quad x \otimes b \to bx$$

is injective and $\phi^{-1}(IB) = I$.

(vi) *ϕ is injective and* $\operatorname{coker} \phi$ *is flat over A.*

Proof.

(i)⇒(ii) follows from 1.2.4.

(ii)⇒(iii) Let $\mathfrak{m}$ be a maximal ideal of A. Suppose $\operatorname{Spec} B \to \operatorname{Spec} A$ is onto. Then there exists a prime ideal $\mathfrak{q}$ of B such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{m}$. Let $\mathfrak{n}$ be a maximal ideal of B containing $\mathfrak{q}$. Then $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(iii)⇒(i) For any maximal ideal $\mathfrak{m}$ of A, let $\mathfrak{n}$ be a maximal ideal of B such that $\phi^{-1}(\mathfrak{n}) = \mathfrak{m}$. We have $B \otimes_A A/\mathfrak{m} \cong B/\mathfrak{m}B$, and $B/\mathfrak{m}B$ has a quotient $B/\mathfrak{n}$ which is nonzero. It follows that $B \otimes_A A/\mathfrak{m} \neq 0$. We then apply 1.2.2.

(i)⇒(iv) Suppose B is faithfully flat over A. To show $M \to M \otimes_A B$ is injective, it suffices to show that the homomorphism

$$M \otimes_A B \to M \otimes_A B \otimes_A B, \quad x \otimes b \to x \otimes 1 \otimes b$$