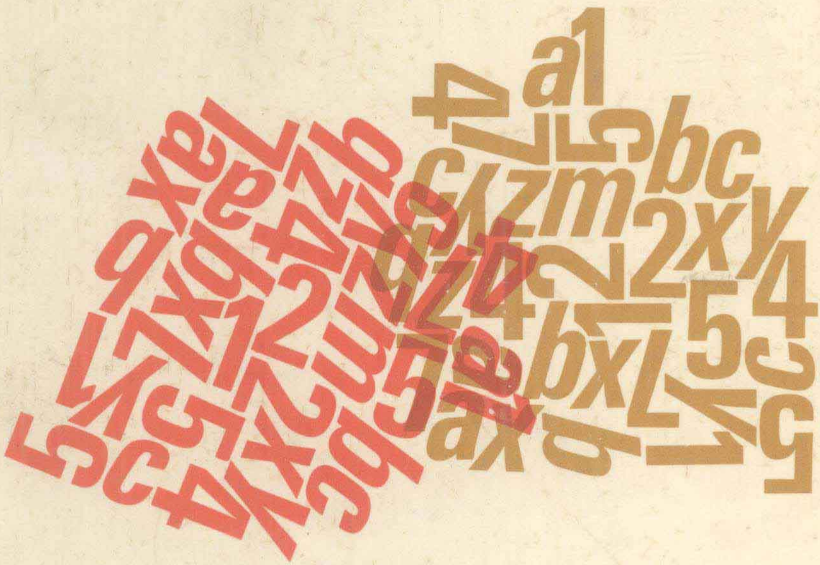
THIRD EDITION

COLLEGE ALGEBRA



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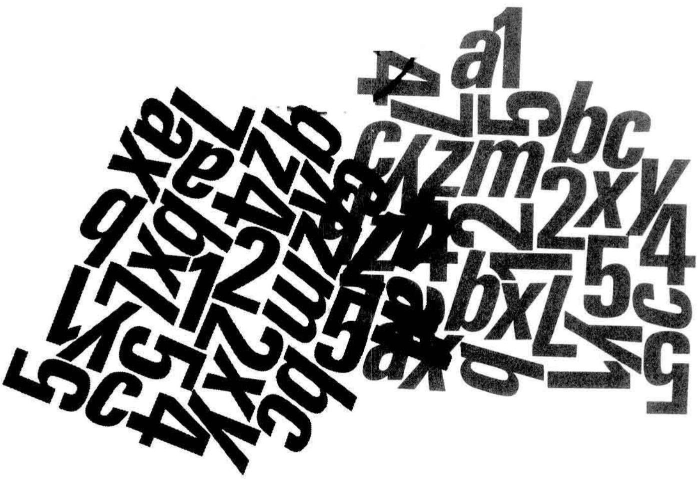

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PREFACE TO THE THIRD EDITION

This edition of *College Algebra*, like the previous ones, has been planned for college freshmen who are not sufficiently undergirded in algebra to begin the study of analytic geometry and calculus. The new edition, however, has been "modernized" through emphasis on the basic concepts, procedures, and terminology. This approach, together with precise definitions and language, makes possible a judicious amount of rigor which in turn facilitates the mastery of the essential techniques employed in the conventional subject matter of elementary algebra.

The concept of a set is introduced in the first chapter. The treatment, though brief, suffices for the use of the concept in the chapters which follow. Set notation and set operations appear at various places where they tend to promote understanding and help with simplifications.

The field axioms of real numbers and certain theorems based on the axioms are treated in Chapter 2. This discussion makes way for justifying the fundamental operations on algebraic expressions. Although students at this stage have been using the axioms and theorems, many do not appreciate the basic nature of the axioms since they do not realize that the theorems stem from the axioms. We have emphasized the dependence of the theorems on the axioms by stating formally all the necessary theorems and referring freely to the axioms in making the proofs. The coverage of the material in the chapter will, of course, depend on the class at hand. As a minimum accomplishment, however, the students, by knowing the axioms and understanding the statements contained in the theorems, should begin to envision elementary algebra as a logical structure, thus gaining some insight into the nature of mathematics.

Several chapters treated at the level of intermediate algebra follow Chapter 2. Then, in Chapter 10, we introduce the order axioms of real numbers and derive certain theorems based on the axioms. This paves the way for solving inequalities. Included in this discussion are inequalities of one and two variables and also included are systems of inequalities.

All the topics of the previous editions, with considerable upgrading in treatment, are taken up in this edition. The new material provides a basis for the improved treatment of the conventional material of the earlier editions.

We list here certain other notable features of the book:

1. Functions and relations are defined as sets of ordered pairs of numbers with the distinction between the two concepts clearly pointed out.

2. A careful discussion is given of operations on equations which lead to equivalent equations, and also on systems of equations which yield equivalent systems.

3. We have achieved a more satisfactory presentation of probability by proceeding from certain axioms and using the idea of sets and subsets.

4. Complex numbers are treated quite adequately. Although the form (a, b) is introduced briefly, the main discussion involves the rectangular form $a + bi$ and the polar form $r(\cos\theta + i\sin\theta)$. Attention is given to the fact that complex numbers obey the field axioms of Chapter 2.

5. An introduction to the algebra of matrices and proofs of the most important properties of determinants of matrices appear in Chapter 16.

6. Logarithms are treated rather fully with emphasis on the theoretical aspects.

7. The exposition throughout is in clear and simple language. Each new topic is amply illustrated with solved problems, these examples having been planned carefully to make evident the principles involved and to obviate any troublesome points. In fact, the organization and exposition are such that the material is largely self-teaching, thus affording more time for the teacher to help with difficult steps and to deepen the student's insight by emphasizing the underlying reasons.

8. As with the text proper, meticulous care has been used in planning the exercises, each of which has an abundance of problems. Answers to the odd-numbered problems are included in the text, and answers to the even-numbered ones are given in the instructor's manual.

9. All the necessary numerical tables—powers and roots of numbers, common logarithms of numbers, and values of trigonometric functions—are bound with the book.

10. The book is flexible enough for the selection of material to accommodate classes of varying degrees of preparation. Some students may need to spend considerable time on Chapters 3 through 8, while others, in a short time, can successfully review these chapters (or even omit them) and have ample time to cover most of the remaining chapters, thus securing a promising foundation for the study of calculus and other areas of mathematics. Although three class meetings a week for a semester is the most common practice, ample material has been provided for students who would be better served by four or five class meetings a week.

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Chapter 1

SETS AND OPERATIONS

1-1 SETS

The idea of a set is a fundamental concept in mathematics. Although a set is classed as an undefined term, we gain an intuitive understanding by describing a set as a specified collection (or aggregate) of objects. Thus the totality of trees in a park, the collection of chairs in a room, and the list of families in a town are examples of sets. Each object of a set is called an *element* or *member* of the set. A set may be specified by listing its elements or by describing the set so that it can be determined if any given object is, or is not, a member of the set.

Many kinds of sets arise in mathematical investigations. We mention a few examples of sets, using a capital letter to stand for each set.

A = the set of all angles having measures between 0° and 90°.
B = the set of vowels in the English alphabet.
C = the set of positive integers less than 7.*
D = the set of all numbers between 3 and 5.
E = the set of counties in Texas.
F = the positive odd integers less than 100.

We note that the elements of the sets B, C, E, and F may be listed. But we cannot list the elements of sets A and D. If the elements of a set can be listed from the first through the last, we say the set is *finite*; otherwise the set is *infinite*. It is customary to specify an infinite set by enclosing a description of the set within braces. Although a finite set can be indicated in this way, it is sometimes more convenient simply to list the elements. We may specify the sets above with the following notation.

$$A = \{\theta | 0° < \theta < 90°\}$$
$$B = \{a, e, i, o, u\}$$
$$C = \{1, 2, 3, 4, 5, 6\}$$
$$D = \{x | 3 < x < 5\}$$

* We assume the existence of the numbers 1, 2, 3, 4, and so on. This set of numbers is called the *counting numbers*, the *natural numbers*, or the *positive integers*.

$E = \{x | x \text{ is a county in Texas}\}$
$F = \{1, 3, 5, 7, \ldots, 99\}$

The letter θ in set A stands for any unspecified member of the set and the symbols $<$ and $>$ mean respectively "is less than" and "is greater than." Hence the set consists of all angles whose measures are greater than 0° and less than 90°. Similarly, x in set D stands for any unspecified member of the set which consists of all numbers greater than 3 and less than 5. The vertical bar in sets A, D, and E may be read "such that," and the three dots in set F indicate that the missing odd integers are members of the set. With these explanations, we see that the members of each set A to F are definitely spelled out.

We express the fact that θ is a member of set A, and x is a member of set D, by writing

$$\theta \in A \quad \text{and} \quad x \in D$$

The symbol $\in$ may be read "is a member of, is an element of," or "belongs to"; and the symbol $\notin$ is read "is not a member of." The letters θ and x, as used here, are variables, as is borne out by the following definition.

Definition 1-1 A symbol, usually a letter, which may stand for any member of a specified set of objects is called a *variable*. If the set has only one member the symbol is called a *constant*.

1-2 RELATED SETS

In this section we shall point out certain ways in which two sets may be related.

Definition 1-2 Two sets A and B are said to be *equal* ($A = B$) if each element of set A is an element of set B and each element of set B is an element of set A.

The sets $A = \{x, y, z\}$ and $B = \{y, z, x\}$, for example, are equal.

Definition 1-3 If it is possible to pair each element of a set A with exactly one element of a set B and each element of B with exactly one element of A, then we say the elements of the sets can be arranged in a *one-to-one correspondence*.

Definition 1-4 Two sets A and B are said to be *equivalent* ($A \leftrightarrow B$) if their elements can be put into a one-to-one correspondence.

According to this definition the infinite set of positive integers and the infinite set of positive even integers are equivalent. The equivalence of the sets $A = \{1, 2, 3, \ldots\}$ and $B = \{2, 4, 6, \ldots\}$ becomes evident from the following method of pairing the elements:

$$(1, 2), (2, 4), (3, 6), (4, 8), (5, 10), \ldots$$

Definition 1-5 If each element of a set A is an element of set B, then A is called a *subset* of B. If A is a subset of B and if B has one or more elements not belonging to A, then A is a *proper* subset of B.

To indicate that A is a subset of B, we write

$$A \subset B$$

which is read "A is contained in B" or "B contains A."

We write here some of the subsets of $\{a, b, c, d\}$.

$$\{a\} \subset \{a, b, c, d\}$$

$$\{b, c\} \subset \{a, b, c, d\}$$

$$\{a, c, d\} \subset \{a, b, c, d\}$$

$$\{a, b, c, d\} \subset \{a, b, c, d\}$$

As illustrated by the last subset appearing here, a set is a subset of itself; this is a consequence of the very definition of a subset.

Thus far we have mentioned sets which contain one or more elements. Conditions may be specified, however, such that a set has no element. For example, the set of blondes in a group of five brunettes has no element. A set which contains no element is called the *null*, or *empty*, set. The symbol $\varnothing$ is used to stand for the empty set. Thus,

$$\{x|x = 1 \text{ and } x = 2\} = \varnothing$$

$$\{x|x \text{ is an integer between 5 and 6}\} = \varnothing$$

Since $\varnothing$ has no element, we can conclude that each element of $\varnothing$ belongs to any set A, and therefore $\varnothing \subset A$. In other words, the empty set is a subset of every set.

EXERCISE 1-1

Enclose within braces (a) a list of the elements of each set and (b) a description of the set.

1. The positive integers less than 7.
2. The positive even integers between 1 and 13.
3. The days of the week.
4. The months of the year having 30 days.
5. The first three presidents of the United States.
6. The thirteen original states of the United States.
7. The books of the Pentateuch.

Tell if the two sets are equal, equivalent, or if one is a subset of the other.

8. $A = \{4, 7, 11\}$, $B = \{7, 4, 11\}$.
9. $C = \{a, e, i, o, u\}$, $D = \{u, v, x, y\}$.

10. $C = \{x|x$ is a number between 1 and 20 and divisible by $4\}$
$D = \{t|t$ is a positive integer less than $20\}$.
11. $S = \{x|x$ is a lady president of the United States$\}$
$T = \{x|x$ is a positive integer less than $1\}$.

12. Given $\{5, 7, 8\} = \{x, 8, 7\}$, find x.
13. Find a set of values for a, b, c such that $\{a, b, c\} = \{0, 1, 4\}$.
14. Write all the subsets of (a) $\{1\}$, (b) $\{u, v\}$.
15. Write the eight subsets of $\{1, 2, 3\}$.
16. If $E = \{2, 5, 8, 11\}$ and $F = \{8, 11, 14, 2\}$, tell which of the following is true: (a) $8 \in E$, (b) $8 \subset F$, (c) $8 \subset E$, (d) $\varnothing \in F$, (e) $\varnothing \subset F$, (f) $\{8\} \in E$, (g) $\{8\} \subset E$, (h) $E \leftrightarrow F$, (i) $\{2, 5, 8\}$ is a proper subset of E.
17. If $A \subset B$ and $B \subset A$, show that A is (or is not) equal to B.
18. By using definitions concerning sets in Sec. 1-2, show that the following are true for sets A, B, and C:
(a) $A = A$ (equality is reflexive).
(b) If $A = B$, then $B = A$ (equality is symmetric).
(c) If $A = B$ and $B = C$, then $A = C$ (equality is transitive).

1-3 OPERATIONS ON SETS

We have seen that new sets, called subsets, can be obtained from a given set. In this section we shall consider operations by which new sets can be obtained from two given sets.

Definition 1-6 The set consisting of the totality of elements under consideration in a particular discussion is called the *universal set*. The universal set is commonly denoted by U.

Definition 1-7 The set of all elements which belong to a given universal set U and do not belong to a given subset A of U is called the *complement* of A.

Denoting the complement of a set A by A', we may write

$$A' = \{x|x \in U \text{ and } x \notin A\}$$

Suppose, for example, that we have under consideration the sets $A = \{a, b, c\}$ and $B = \{c, d, f, g\}$. Then we could choose $U = \{a, b, c, d, f, g\}$ and have

$$A' = \{d, f, g\} \quad \text{and} \quad B' = \{a, b\}$$

Definition 1-8 The *union* of two sets A and B, denoted by $A \cup B$, is defined to be the set composed of all elements which belong to A or to B or to both A and B.

It follows from this definition that if $x \in A$ then $x \in A \cup B$ and if $x \in B$, then $x \in A \cup B$. Hence we have

$$A \cup B = \{x|x \in A \text{ or } x \in B\}$$

Definition 1-9 The *intersection* of two sets A and B, denoted by $A \cap B$, is defined to be the set consisting of all elements which belong to A and also belong to B.

We express this definition symbolically by writing

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

EXAMPLE 1. If $A = \{u, v, w, x, y\}$ and $B = \{a, v, w, y\}$, then

$$A \cup B = \{u, v, w, x, y, a\}$$

and

$$A \cap B = \{v, w, y\}$$

EXAMPLE 2. If $C = \{5, 7, 11\}$ and $D = \{\text{sun, moon}\}$, then

$$C \cup D = \{5, 7, 11, \text{sun, moon}\}$$

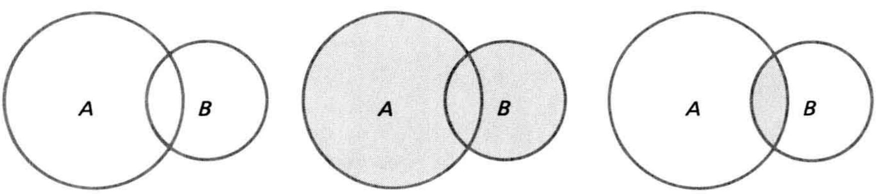
and

$$C \cap D = \varnothing$$

Since C and D have no common element, their intersection is the empty set $\varnothing$.

Definition 1-10 Two sets which have no common element are called *disjoint sets*.

Figures 1-1 and 1-2 furnish pictorial illustrations of the union and intersection of two sets. We let A stand for the set of all points inside the larger circle and B the set of all points inside the smaller circle. Pictures of this kind are called Venn diagrams in honor of the English logician John Venn (1834–1883).

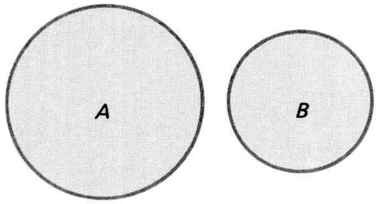


$A \cup B$ shaded

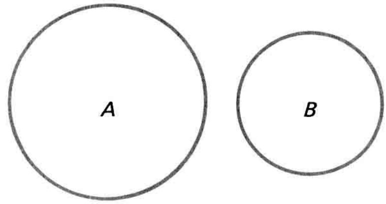
$A \cap B$ shaded

Fig. 1-1

Definition 1-11 If x and y denote two objects (alike or different) with x specified as the first object and y the second object, then the symbol (x, y) is called an *ordered pair*.



$A \cup B$ shaded



$A \cap B = \varnothing$

Fig. 1-2

Definition 1-12 If X and Y are sets, the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$ is called the *product set*, or Cartesian product, of X and Y; or, in symbols,

$$X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$$

EXAMPLE 3. If $X = \{a, b\}$ and $Y = \{1, 2, 3\}$, the product set is

$$X \times Y = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

EXAMPLE 4. If $S = \{(a, b)\}$, then

$$S \times S = \{(a, a), (a, b), (b, a), (b, b)\}$$

EXERCISE 1-2

If $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{0, 1, 3, 4\}$, and $C = \{4, 5, 6, 7\}$, enclose within braces the elements of the following sets.

1. $A \cup B$ **2.** $A \cap B$ **3.** $A \cup C$ **4.** $B \cap B$
5. $C \cup C$ **6.** $A \cup \varnothing$ **7.** $B \cap \varnothing$ **8.** $A \cup (B \cup C)$
9. $A \cup (B \cap C)$ **10.** $A \cap (B \cap C)$ **11.** $A \cap (B \cup C)$ **12.** $A \cup (B \cap \varnothing)$

If $A = \{p, q, r, s\}$, $B = \{r, s, t, u\}$, and $C = \{t, u, v, w\}$, determine if the following equations are true by examining each member of the equations.

13. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
14. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
15. $A \cap (\varnothing \cup C) = (A \cup B) \cap (A \cup C)$

Prove that if A and B are any sets, then each of the following equations is true.

16. $A \cup A = A$ **17.** $A \cap A = A$
18. $A \cap B = B \cap A$ **19.** $A \cup B = B \cup A$

Draw Venn diagrams to illustrate the following theorems.

20. *Theorem.* If $A \subset B$, then $B \cup A = B$.
21. *Theorem.* If $B \cup A = B$, then $A \subset B$.
22. *Theorem.* If $A \cap B = A$, then $A \subset B$.
23. *Theorem.* If $A \subset B$, then $A \cap B = A$.

24. If $X = \{1\}$ and $Y = \{2, 3\}$, find $X \times Y$ and $Y \times X$.
25. If $P = \{1, 2\}$ and $Q = \{2, 3\}$, find $P \times Q$ and $Q \times P$.
26. If $S = \{a, b, c\}$ and $T = \{1, 2\}$, find $S \times T$.