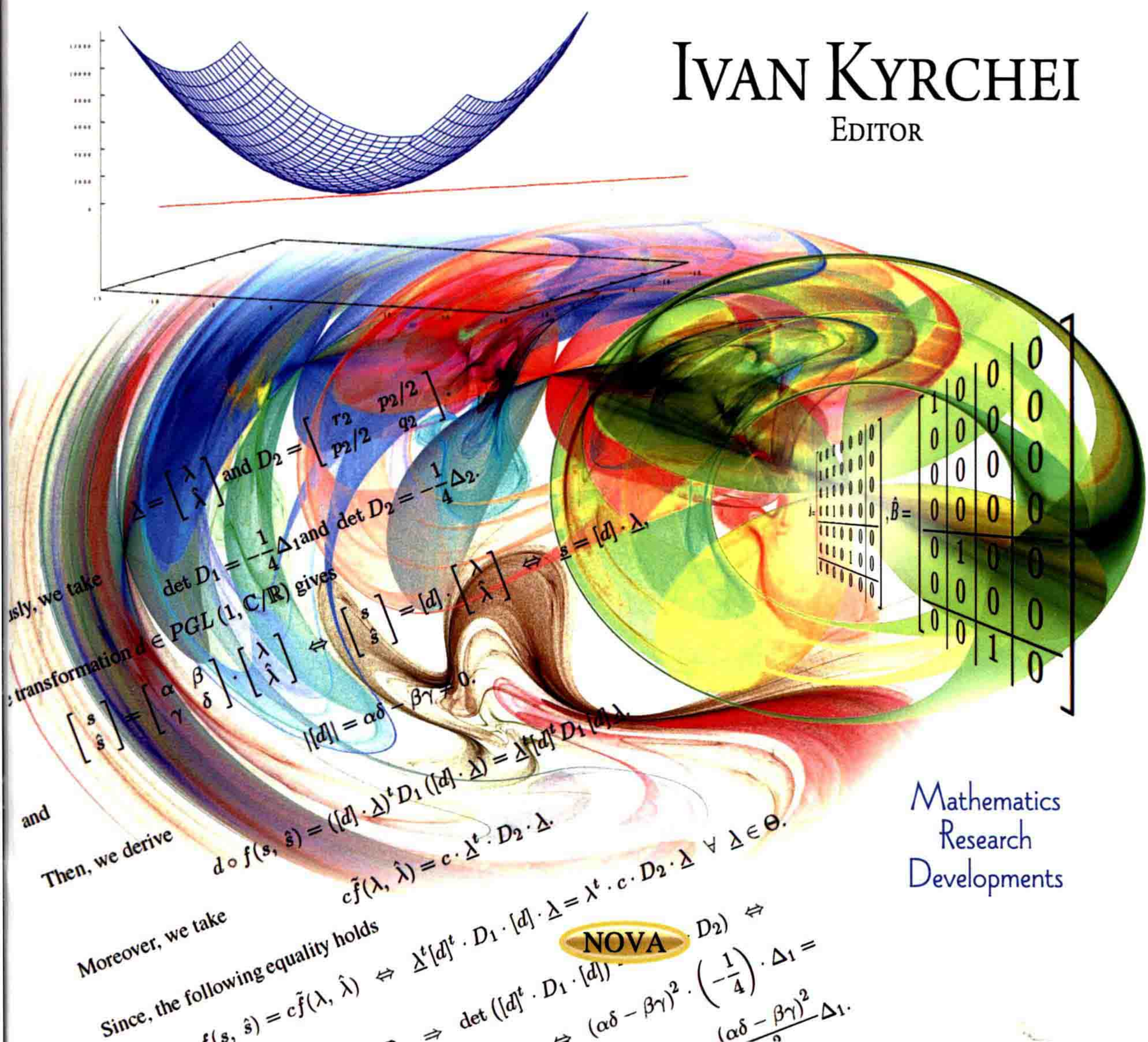
Advances in Linear Algebra Research

Ivan Kyrchei
Editor



Mathematics Research Developments

NOVA

MATHEMATICS RESEARCH DEVELOPMENTS

ADVANCES IN LINEAR ALGEBRA RESEARCH

IVAN KYRCHEI
EDITOR

nova publishers
New York

LIBRARY OF CONGRESS CATALOGING-IN-PUBLICATION DATA

Advances in linear algebra research / Ivan Kyrchei (National Academy of Sciences of Ukraine), editor.
pages cm. -- (Mathematics research developments)
Includes bibliographical references and index.
ISBN 978-1-63463-565-3 (hardcover)
1. Algebras, Linear. I. Kyrchei, Ivan, editor.
QA184.2.A38 2015
512'.5--dc23

2014043171

Published by Nova Science Publishers, Inc. † New York

MATHEMATICS RESEARCH DEVELOPMENTS

ADVANCES IN LINEAR ALGEBRA RESEARCH

MATHEMATICS RESEARCH DEVELOPMENTS

Additional books in this series can be found on Nova's website under the Series tab.

Additional e-books in this series can be found on Nova's website under the e-book tab.

PREFACE

This book presents original studies on the leading edge of linear algebra. Each chapter has been carefully selected in an attempt to present substantial research results across a broad spectrum. The main goal of Chapter One is to define and investigate the restricted generalized inverses corresponding to minimization of constrained quadratic form. As stated in Chapter Two, in systems and control theory, Linear Time Invariant (LTI) descriptor (Differential-Algebraic) systems are intimately related to the matrix pencil theory. A review of the most interesting properties of the Projective Equivalence and the Extended Hermite Equivalence classes is presented in the chapter. New determinantal representations of generalized inverse matrices based on their limit representations are introduced in Chapter Three. Using the obtained analogues of the adjoint matrix, Cramer's rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems have been obtained in the chapter. In Chapter Four, a very interesting application of linear algebra of commutative rings to systems theory, is explored. Chapter Five gives a comprehensive investigation to behaviors of a general Hermitian quadratic matrix-valued function by using ranks and inertias of matrices. In Chapter Six, the theory of triangular matrices (tables) is introduced. The main "characters" of the chapter are special triangular tables (which will be called triangular matrices) and their functions paradeterminants and parapermanents. The aim of Chapter Seven is to present the latest developments in iterative methods for solving linear matrix equations. The problems of existence of common eigenvectors and simultaneous triangularization of a pair of matrices over a principal ideal domain with quadratic minimal polynomials are investigated in Chapter Eight. Two approaches to define a noncommutative determinant (a determinant of a matrix with noncommutative elements) are considered in Chapter Nine. The last, Chapter 10, is an example of how the methods of linear algebra are used in natural sciences, particularly in chemistry. In this chapter, it is shown that in a First Order Chemical Kinetics Mechanisms matrix, all columns add to zero, all the diagonal elements are non-positive and all the other matrix entries are non-negative. As a result of this particular structure, the Gershgorin Circles Theorem can be applied to show that all the eigenvalues are negative or zero.

Minimization of a quadratic form $\langle x, Tx\rangle + \langle p, x\rangle + a$ under constraints defined by a linear system is a common optimization problem. In Chapter 1, it is assumed that the operator T is symmetric positive definite or positive semidefinite. Several extensions to different sets of linear matrix constraints are investigated. Solutions of this problem may be given using the Moore-Penrose inverse and/or the Drazin inverse. In addition, several new classes of generalized inverses are defined minimizing the seminorm defined by the quadratic forms, depending on the matrix equation that is used as a constraint.

A number of possibilities for further investigation are considered.

In systems and control theory, Linear Time Invariant (LTI) descriptor (Differential-Algebraic) systems are intimately related to the matrix pencil theory. Actually, a large number of systems are reduced to the study of differential (difference) systems $S(F,G)$ of the form:

$$S(F,G): F\dot{x}(t) = Gx(t) \text{ (or the dual } Fx = G\dot{x}(t)),$$

and

$$S(F,G): Fx_{k+1} = Gx_k \text{ (or the dual } Fx_k = Gx_{k+1}), \ F, G \in \mathbb{C}^{m \times n}$$

and their properties can be characterized by the homogeneous pencil $sF - \hat{s}G$. An essential problem in matrix pencil theory is the study of invariants of $sF - \hat{s}G$ under the *bilinear strict equivalence*. This problem is equivalent to the study of complete *Projective Equivalence* (PE), $\mathcal{E}_{\mathcal{P}}$, defined on the set $\mathbb{C}_r$ of complex homogeneous binary polynomials of fixed homogeneous degree r. For a $f(s,\hat{s}) \in \mathbb{C}_r$, the study of invariants of the PE class $\mathcal{E}_{\mathcal{P}}$ is reduced to a study of invariants of matrices of the set $\mathbb{C}^{k\times 2}$ (for $k \geqslant 3$ with all 2×2-minors non-zero) under the *Extended Hermite Equivalence* (EHE), $\mathcal{E}_{rh}$. In Chapter 2, the authors present a review of the most interesting properties of the PE and the EHE classes. Moreover, the appropriate projective transformation $d \in RGL(1, \mathbb{C}/\mathbb{R})$ is provided analytically ([1]).

By a generalized inverse of a given matrix, the authors mean a matrix that exists for a larger class of matrices than the nonsingular matrices, that has some of the properties of the usual inverse, and that agrees with inverse when given matrix happens to be nonsingular. In theory, there are many different generalized inverses that exist. The authors shall consider the Moore Penrose, weighted Moore-Penrose, Drazin and weighted Drazin inverses.

New determinantal representations of these generalized inverse based on their limit representations are introduced in Chapter 3. Application of this new method allows us to obtain analogues classical adjoint matrix. Using the obtained analogues of the adjoint matrix, the authors get Cramer's rules for the least squares solution with the minimum norm and for the Drazin inverse solution of singular linear systems. Cramer's rules for the minimum norm least squares solutions and the Drazin inverse solutions of the matrix equations $\mathbf{AX} = \mathbf{D}$, $\mathbf{XB} = \mathbf{D}$ and $\mathbf{AXB} = \mathbf{D}$ are also obtained, where $\mathbf{A}$, $\mathbf{B}$ can be singular matrices of appropriate size. Finally, the authors derive determinantal representations of solutions of the differential matrix equations, $\mathbf{X}' + \mathbf{AX} = \mathbf{B}$ and $\mathbf{X}' + \mathbf{XA} = \mathbf{B}$, where the matrix $\mathbf{A}$ is singular.

Many physical systems in science and engineering can be described at time t in terms of an n-dimensional state vector $x(t)$ and an m-dimensional input vector $u(t)$, governed by an evolution equation of the form $x'(t) = A \cdot x(t) + B \cdot u(t)$, if the time is continuous, or $x(t+1) = A \cdot x(t) + B \cdot u(t)$ in the discrete case. Thus, the system is completely described by the pair of matrices (A, B) of sizes $n \times n$ and $n \times m$ respectively.

In two instances feedback is used to modify the structure of a given system (A, B): first, A can be replaced by $A + BF$, with some characteristic polynomial that ensures stability of the new system $(A + BF, B)$; and second, combining changes of bases with a feedback action $A \mapsto A + BF$ one obtains an equivalent system with a simpler structure.

Given a system (A, B), let (A, B) denote the set of states reachable at finite time when starting with initial condition $x(0) = 0$ and varying $u(t)$, i.e., (A, B) is the right image of the matrix $[B|AB|A^2B|\cdots]$. Also, let $Pols(A, B)$ denote the set of characteristic polynomials of all possible matrices $A + BF$, as F varies.

Classically, (A, B) have their entries in the field of real or complex numbers, but the concept of discrete-time system is generalized to matrix pairs with coefficients in an arbitrary commutative ring R. Therefore, techniques from Linear Algebra over commutative rings are needed.

In Chapter 4, the following problems are studied and solved when R is a commutative von Neumann regular ring:

- A canonical form is obtained for the feedback equivalence of systems (combination of basis changes with a feedback action).
- Given a system (A, B), it is proved that there exist a matrix F and a vector u such that the single-input system $(A + BF, Bu)$ has the same reachable states and the same assignable polynomials as the original system, i.e. $(A + BF, Bu) = (A, B)$ and $Pols(A + BF, Bu) = Pols(A, B)$.

Chapter 5 gives a comprehensive investigation to behaviors of a general Hermitian quadratic matrix-valued function

$$\phi(X) = (\, AXB + C\,)M(\, AXB + C)^* + D$$

by using ranks and inertias of matrices. The author first establishes a group of analytical formulas for calculating the global maximal and minimal ranks and inertias of $\phi(X)$. Based on the formulas, the author derives necessary and sufficient conditions for $\phi(X)$ to be a positive definite, positive semi-definite, negative definite, negative semi-definite function, respectively, and then solves two optimization problems of finding two matrices $\widehat{X}$ or $\widetilde{X}$ such that $\phi(X) \succcurlyeq \phi(\widehat{X})$ and $\phi(X) \preccurlyeq \phi(\widetilde{X})$ hold for all X, respectively. As extensions, the author considers definiteness and optimization problems in the Löwner sense of the following two types of multiple Hermitian quadratic matrix-valued function

$$\phi(\, X_1, \ldots, X_k\,) = \left(\sum_{i=1}^{k} A_iX_iB_i + C \right) M \left(\sum_{i=1}^{k} A_iX_iB_i + C \right)^* + D,$$

$$\psi(\, X_1, \ldots, X_k\,) = \sum_{i=1}^{k} (\, A_iX_iB_i + C_i\,)M_i(\, A_iX_iB_i + C_i\,)^* + D.$$

Some open problems on algebraic properties of these matrix-valued functions are mentioned at the end of Chapter 5.

In Chapter 6, the author considers elements of linear algebra based on triangular tables with entries in some number field and their functions, analogical to the classical notions of a matrix, determinant and permanent. Some properties are investigated and applications in various areas of mathematics are given.

The aim of Chapter 7 is to present the latest developments in iterative methods for solving linear matrix equations. The iterative methods are obtained by extending the methods presented to solve the linear system $Ax = b$. Numerical examples are investigated to confirm the efficiency of the methods.

The problems of existence of common eigenvectors and simultaneous triangularization of a pair of matrices over a principal ideal domain with quadratic minimal polynomials are investigated in Chapter 8. The necessary and sufficient conditions of simultaneous triangularization of a pair of matrices with quadratic minimal polynomials are obtained. As a result, the approach offered provides the necessary and sufficient conditions of simultaneous triangularization of pairs of idempotent matrices and pairs of involutory matrices over a principal ideal domain.

Since product of quaternions is noncommutative, there is a problem how to determine a determinant of a matrix with noncommutative elements (it's called a noncommutative determinant). The authors consider two approaches to define a noncommutative determinant. Primarily, there are row – column determinants that are an extension of the classical definition of the determinant; however, the authors assume predetermined order of elements in each of the terms of the determinant. In Chapter 9, the authors extend the concept of an immanant (permanent, determinant) to a split quaternion algebra using methods of the theory of the row and column determinants.

Properties of the determinant of a Hermitian matrix are established. Based on these properties, analogs of the classical adjont matrix over a quaternion skew field have been obtained. As a result, the authors have a solution of a system of linear equations over a quaternion division algebra according to Cramer's rule by using row–column determinants.

Quasideterminants appeared from the analysis of the procedure of a matrix inversion. By using quasideterminants, solving of a system of linear equations over a quaternion division algebra is similar to the Gauss elimination method.

The common feature in definition of row and column determinants and quasideterminants is that the authors have not one determinant of a quadratic matrix of order n with noncommutative entries, but certain set (there are n^2 quasideterminants, n row determinants, and n column determinants). The authors have obtained a relation of row-column determinants with quasideterminants of a matrix over a quaternion division algebra.

First order chemical reaction mechanisms are modeled through Ordinary Differential Equations (O.D.E.) systems of the form: , being the chemical species concentrations vector, its time derivative, and the associated system matrix.

A typical example of these reactions, which involves two species, is the Mutarotation of Glucose, which has a corresponding matrix with a null eigenvalue whereas the other one is negative.

A very simple example with three chemical compoundsis grape juice, when it is converted into wine and then transformed into vinegar. A more complicated example,also involving three species, is the adsorption of Carbon Dioxide over Platinum surfaces. Although, in these examples the chemical mechanisms are very different, in both cases the O.D.E. system matrix has two negative eigenvalues and the other one is zero. Consequently, in all these cases that involve two or three chemical species, solutions show a weak stability (i.e., they are stable but not asymptotically). This fact implies that small errors due to measurements in the initial concentrations will remain bounded, but they do not tend to vanish as the reaction proceeds.

In order to know if these results can be extended or not to other chemical mechanisms, a possible general result is studied through an inverse modeling approach, like in previous papers. For this purpose, theoretical mechanisms involving two or more species are proposed and a general type of matrices - so-called First Order Chemical Kinetics Mechanisms (F.O.C.K.M.) matrices - is studied from the eigenvalues and eigenvectors view point.

Chapter 10 shows that in an F.O.C.K.M. matrix all columns add to zero, all the diagonal elements are non-positive and all the other matrix entries are non-negative. Because of this particular structure, the Gershgorin Circles Theorem can be applied to show that all the eigenvalues are negative or zero. Moreover, it can be proved that in the case of the null eigenvalues - under certain conditions - algebraic and geometric multiplicities give the same number.

As an application of these results, several conclusions about the stability of the O.D.E. solutions are obtained for these chemical reactions, and its consequences on the propagation of concentrations and/or surface concentration measurement errors, are analyzed.

CONTENTS

In: Advances in Linear Algebra Research
Editor: Ivan Kyrchei, pp. 1-55
ISBN: 978-1-63463-565-3

Chapter 1

MINIMIZATION OF QUADRATIC FORMS AND GENERALIZED INVERSES

Predrag S. Stanimirović[1,*]***, Dimitrios Pappas***[2,†] ***and Vasilios N. Katsikis***[3,‡]
[1]University of Niš, Faculty of Sciences and Mathematics, Niš, Serbia
[2]Athens University of Economics and Business
Department of Statistics, Athens, Greece
[3]National and Kapodistrian University of Athens, Department of Economics
Division of Mathematics and Informatics, Athens, Greece

Abstract

Minimization of a quadratic form $\langle x, Tx\rangle + \langle p, x\rangle + a$ under constraints defined by a linear system is a common optimization problem. It is assumed that the operator T is symmetric positive definite or positive semidefinite. Several extensions to different sets of linear matrix constraints are investigated. Solutions of this problem may be given using the Moore-Penrose inverse and/or the Drazin inverse. In addition, several new classes of generalized inverses are defined minimizing the seminorm defined by the quadratic forms, depending on the matrix equation that is used as a constraint.

A number of possibilities for further investigation are considered.

Keywords: Quadratic functional, quadratic optimization, generalized inverse, Moore-Penrose inverse, Drazin inverse, outer inverse, system of linear equations, matrix equation, generalized inverse solution, Drazin inverse solution

AMS Subject Classification: 90C20, 15A09, 15A24, 11E04, 47N10

1. Introduction

It is necessary to mention several common and usual notations. By $\mathbb{R}^{m\times n}$ (resp. $\mathbb{C}^{m\times n}$) we denote the space of all real (resp. complex) matrices of dimension $m \times n$. If $A \in$

*E-mail address: pecko@pmf.ni.ac.rs
†E-mail address: pappdimitris@gmail.com
‡E-mail address: vaskatsikis@econ.uoa.gr

$\mathbb{R}^{m\times n}$ (resp. $\mathbb{C}^{m\times n}$), by $A^T \in \mathbb{R}^{n\times m}$ (resp. $A^* \in \mathbb{R}^{n\times m}$) is denoted the transpose (resp. conjugate and transpose) matrix of A. As it is usual, by $\mathcal{N}(A)$ we denote the null-space of A, by $\mathcal{R}(A)$ the range of A, and $\text{ind}(A)$ will denote the index of the matrix A.

1.1. Quadratic Functions, Optimization and Quadratic Forms

Definition 1.1. *A square matrix $A \in \mathbb{C}^{n\times n}$ (resp. $A \in \mathbb{R}^{n\times n}$) is:*

1) *Hermitian (Symmetric) matrix if $A^* = A$ ($A^T = A$),*
2) *normal, if $A^*A = AA^*$ ($A^TA = AA^T$),*
3) *lower-triangular, if $a_{ij} = 0$ for $i < j$,*
4) *upper-triangular, if $a_{ij} = 0$ for $i > j$,*
5) *positive semi-definite, if $\text{Re}\,(x^*Ax) \geq 0$ for all $x \in \mathbb{C}^{n\times 1}$. Additionally, if it holds $\text{Re}\,(x^*Ax) > 0$ for all $x \in \mathbb{C}^{n\times 1} \setminus \{0\}$, then the matrix A is positive definite.*

6) *Unitary (resp. orthogonal) matrix A is a square matrix whose inverse is equal to its conjugate transpose (resp. transpose), $A^{-1} = A^*$ (resp. $A^{-1} = A^T$).*

Definition 1.2. *Let $A \in \mathbb{C}^{m\times n}$. A real or complex scalar λ which satisfies the following equation*

$$Ax = \lambda x, \quad \text{i.e.,} \quad (A - \lambda I)x = 0,$$

is called an eigenvalue of A, and x is called an eigenvector of A corresponding to λ.

The eigenvalues and eigenvectors of a matrix play a very important role in matrix theory. They represent a tool which enables to understand the structure of a matrix. For example, if a given square matrix of complex numbers is self-adjoint, then there exist basis of $\mathbb{C}^m$ and $\mathbb{C}^n$, consisting of distinct eigenvectors of A, with respect to which the matrix A can be represented as a diagonal matrix. But, in the general case, not every matrix has enough distinct eigenvectors to enable its diagonal decomposition. The following definition, given as a generalization of the previous one, is useful to resolve this problem.

Definition 1.3. *Let $A \in \mathbb{C}^{m\times n}$ and λ is an eigenvalue of A. A vector x is called generalized eigenvector of A of grade p corresponding to λ, or λ-vector of A of grade p, if it satisfies the following equation*

$$(A - \lambda I)^p x = 0.$$

Namely, for each square matrix there exists a basis composed of generalized eigenvectors with respect to which, a matrix can be represented in the Jordan form. Corresponding statement is stated in the following proposition.

Proposition 1.1. [1] *(The Jordan decomposition). Let the matrix $A \in \mathbb{C}^{n\times n}$ has p distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then A is similar to a block diagonal matrix J with Jordan blocks on its diagonal, i.e., there exists a nonsingular matrix P which satisfies*

$$AP = PJ = \begin{bmatrix} J_{k_1}(\lambda_1) & 0 & \ldots & 0 \\ 0 & J_{k_2}(\lambda_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & J_k(\lambda_p) \end{bmatrix},$$

where the Jordan blocks are defined by

$$J_{k_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \lambda_i & 1 \end{bmatrix}$$

and the matrix J is unique up to a rearrangement of its blocks.

The following Proposition 1.2 gives us an alternative way to obtain even simpler decomposition of the matrix A, than the one given with the Jordan decomposition, but with respect to a different basis of $\mathbb{C}^n$. This decomposition is known as the Singular Value Decomposition (SVD shortly) and it is based on the notion of singular values, given in Definition 1.4.

Definition 1.4. *Let $A \in \mathbb{C}^{m\times n}$ and $\{\lambda_1, \dots, \lambda_p\}$ be the nonzero eigenvalues of AA^*. The singular values of A, denoted by $\sigma_i(A)$, $i = 1, \dots, p$ are defined by*

$$\sigma_i(A) = \sqrt{\lambda_i}, \quad i = 1, \dots, p.$$

Proposition 1.2. *(Singular value decomposition)* [1] *Let $A \in \mathbb{R}^{m\times n}$ be a matrix with singular values $\{\sigma_1, \dots, \sigma_r\}$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that*

$$A = U\Sigma V^T,$$

where Σ is a nonsquare diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \vdots & 0 \\ & \ddots & & \vdots & 0 \\ & & \sigma_r & \vdots & \\ \dots & \dots & \dots & \dots & \dots \\ & 0 & & \vdots & 0 \end{bmatrix}, \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r.$$

A square matrix T of the order n is symmetric and positive semidefinite (abbreviated SPSD and denoted by $T \succeq 0$) if

$$v^T T v \geq 0 \text{ for all } v \in \mathbb{R}^n.$$

T is symmetric and positive definite (abbreviated SPD and denoted by $T \succ 0$) if

$$v^T T v > 0 \text{ for all } v \in \mathbb{R}^n, v \neq 0.$$

Recall that a symmetric matrix T is positive definite if and only if all its eigenvalues are nonnegative. The corresponding minimization problem, stated originally in linear algebra and frequently used in many scientific areas, is to minimize the quadratic form

$$\frac{1}{2}x^T T x + p^T x + a = \frac{1}{2}\langle x, Tx\rangle + p^T x + a \tag{1.1}$$

with respect to unknown vector $x \in \mathbb{R}^n$. Here T is a square positive definite matrix of the order n, $p \in \mathbb{R}^n$ is a vector of length n and a is a real scalar. Optimization problem (1.1) is called an *unconstrained quadratic optimization problem.*

Let $x, p, a \in \mathbb{R}^n$ are real vectors and T is a symmetric $n \times n$ matrix. The linearly constrained quadratic programming problem can be formulated as follows (see, for example, [2]):
Minimize the goal function (1.1) subject to one or more inequality and/or equality constraints defined by two $n \times n$ matrices A, E and two n-dimensional vectors b, d:

$$Ax \leq b$$
$$Ex = d.$$

Notice that in the general Quadratic Programming model (QP model shortly) we can always presume that T is a symmetric matrix. Indeed, because

$$x^T Tx = \frac{1}{2}x^T(T + T^T)x.$$

it is possible to replace T by the symmetric matrix $\tilde{T} = \frac{1}{2}(T + T^T)$.

Proposition 1.3. *An arbitrary symmetric matrix T is diagonalizable in the general form $T = RDR^T$, where R is an orthonormal matrix, the columns of R are an orthonormal basis of eigenvectors of T, and D is a diagonal matrix of the eigenvalues of T.*

Proposition 1.4. *If $T \in \mathbb{R}^{n\times n}$ is symmetric PSD matrix, then the following statements are equivalent:*
1) $T = MM^T$, *for an appropriate matrix M of the order $M \in \mathbb{R}^{n\times k}$, $k \geq 1$.*
2) $v^T Tv \geq 0$ *for all* $v \in \mathbb{R}^n, v \neq 0$.
3) *There exist vectors v_i, $i = 1, \ldots, n \in \mathbb{R}^k$ (for some $k \geq 1$) such that $T_{ij} = v_i^T v_j$ for all $i, j = 1, \ldots, n$. The vectors v_i, $i = 1, \ldots, n$, are called a Gram representation of T.*
4) *All principal minors of T are non-negative.*

Proposition 1.5. *Let $T \in \mathbb{C}^{n\times n}$ is symmetric. Then $T \succeq 0$ and it is nonsingular if and only if $T \succ 0$.*

Quadratic forms have played a significant role in the history of mathematics in both the finite and infinite dimensional cases. A number of authors have studied problems on minimizing (or maximizing) quadratic forms under various constraints such as vectors constrained to lie within the unit simplex (see Broom [3]), and the minimization of a more general case of a quadratic form defined in a finite-dimensional real Euclidean space under linear constraints (see e.g. La Cruz [4], Manherz and Hakimi [5]), with many applications in network analysis and control theory (for more on this subject, see also [6, 7]). In a classical book on optimization theory, Luenberger [8], presented similar optimization problems for both finite and infinite dimensions. Quadratic problems are very important cases in both constrained and non-constrained optimization theory, and they find application in many different areas. First of all, quadratic forms are simple to be described and analyzed, and thus by their investigation, it is convenient to explain the convergence characteristics