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Towards a Theory of Geometric Graphs

János Pach
Editor

American Mathematical Society

The early development of graph theory was heavily motivated and influenced by topological and geometric themes, such as the Königsberg Bridge Problem, Euler's Polyhedral Formula, or Kuratowski's characterization of planar graphs. In 1936, when Dénes König published his classical *Theory of Finite and Infinite Graphs*, the first book ever written on the subject, he stressed this connection by adding the subtitle *Combinatorial Topology of Systems of Segments*. He wanted to emphasize that the subject of his investigations was very *concrete*: planar figures consisting of points connected by straight-line segments. However, in the second half of the twentieth century, graph theoretical research took an interesting turn. In the most popular and most rapidly growing areas (the theory of random graphs, Ramsey theory, extremal graph theory, algebraic graph theory, etc.), graphs were considered as *abstract* binary relations rather than geometric objects. Many of the powerful techniques developed in these fields have been successfully applied in other areas of mathematics. However, the same methods were often incapable of providing satisfactory answers to questions arising in geometric applications.

In the spirit of König, *geometric graph theory* focuses on combinatorial and geometric properties of graphs drawn in the plane by straight-line edges (or more generally, by edges represented by simple Jordan arcs). It is an emerging discipline that abounds in open problems, but it has already yielded some striking results which have proved instrumental in the solution of several basic problems in combinatorial and computational geometry. The present volume is a careful selection of 25 invited and thoroughly refereed papers, reporting about important recent discoveries on the way *Towards a Theory of Geometric Graphs*.

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Towards a Theory of Geometric Graphs

Paul Turán (1910–1976)

Paul Erdős (1913–1996)

To the memories of Paul Erdős and Paul Turán

Preface

The early development of graph theory was heavily motivated and influenced by topological and geometric themes, such as the Königsberg Bridge Problem, Euler's Polyhedral Formula, or Kuratowski's characterization of planar graphs. When in 1936 Dénes König published his classical *Theorie der endlichen und unendlichen Graphen*, the first book ever written on the subject, he stressed this connection by adding the subtitle *Kombinatorische Topologie der Streckenkomplexe* (Combinatorial Topology of Systems of Segments). He wanted to emphasize that the subject of his investigations was very *concrete*: planar figures consisting of points connected by straight-line segments. However, in the second half of the twentieth century, graph theoretical research took an interesting turn. In the most popular, most rapidly growing areas (the theory of random graphs, Ramsey theory, extremal graph theory, algebraic graph theory, etc.), graphs were considered *abstract* binary relations rather than geometric objects. Many of the powerful techniques developed in these fields have been successfully applied in other areas of mathematics, including geometry.

In the past two decades we have witnessed a renaissance of geometry that has permeated virtually every part of mathematics. Graph theory has been no exception. Topological and geometric methods have proved to be instrumental in the solution of many important problems in graph and hypergraph theory. Meanwhile, a number of newly emerging fields of computer science have served as rich sources of exciting open geometric questions. These include computational geometry, robotics, pattern recognition, computerized tomography, VLSI design, graph drawing, computer graphics, computer-aided design (CAD), and geometric information systems (GIS), to mention only a few. Many of the questions arising were related to systems of segments in the plane or to simplices in higher dimensions. Some of them looked very familiar: they had been asked in different contexts much earlier by Paul Erdős, Paul Turán, Micha Perles, John Conway, and others. For instance, Erdős's notoriously difficult questions on the distribution of distances determined by finite point sets in Euclidean spaces boil down to problems about incidences between points and lines, circles, spheres, and other geometric objects (see the survey of Pach and Sharir in this volume). Similar questions turned out to play an important role in robotics. To solve Turán's Brick Factory Problem [**T77**], one has to find a drawing of the complete bipartite graph that minimizes the number of crossings between the edges. The "crossing number" of a graph is a central notion in VLSI design: it is closely related to the minimum area of a chip realizing the same network. According to Conway's Thrackle Conjecture [**W69**], if a graph G can be drawn in the plane so that any pair of its edges cross precisely once (where a common endpoint is counted as a crossing), then G has at most as many edges as vertices. Perles found a very elegant proof for straight-line drawings, and generalized the question in other directions.

For lack of better understanding of the "Combinatorial Topology of Systems of Segments," the traditional methods of graph theory have rarely provided satisfactory answers to questions of the above type. One can often associate a graph or hypergraph to the structure under investigation and observe that it satisfies certain (sometimes fairly simple) geometric conditions. Then one can completely forget about geometry, and finish the proof by establishing a theorem on *abstract* graphs with the given properties. A simple prototype of this approach is the following argument of Erdős [**E46**], which proves that any set P of n points in the plane contains at most $O(n^{3/2})$ point pairs whose distance is one. Connect two elements of P by an edge if they are at unit distance. Since two (unit) circles have at most two points in common, the resulting graph has no $K_{2,3}$ as a subgraph. ($K_{2,3}$ is the complete bipartite graph with 2 and 3 vertices in its classes.) The result now follows from the fact [**KST54**] that the number of edges of any (abstract) $K_{2,3}$-free graph with n vertices is at most $O(n^{3/2})$. This last statement is asymptotically sharp, but the geometric result is not. Exploring further geometric properties, for instance, the structure of crossings, it can be shown [**SSzT84,Sz97**] that the number of unit distance pairs determined by n points is $O(n^{4/3})$. Even this bound is probably very far from being best possible.

A *geometric graph* is a graph drawn by straight-line edges in the plane. It seems that to tackle many problems arising in combinatorial and computational geometry and in the application areas mentioned above, one has to study *geometric* graphs satisfying certain relevant *geometric* conditions. We illustrate this phenomenon by Lovász' proof [**L71**] showing that the number of "halving lines" determined by an n-element point set P in general position in the plane is $O(n^{3/2})$. Suppose n is even. A line ℓ connecting two elements $p, q \in P$ is called a halving line if either of the open half-planes bounded by ℓ contains precisely $\frac{n}{2} - 1$ other points of P. Define a geometric graph G on the vertex set P by connecting $p, q \in P$ with a straight-line segment (edge) if and only if they induce a halving line. It is not hard to argue that any line can cross at most n edges of G. On the other hand, every geometric graph on n vertices that satisfies this condition has $O(n^{3/2})$ edges. In fact, much more is true: Ajtai, Chvátal, Newborn, Szemerédi [**ACN82**] and, independently, Leighton [**L83**] proved that the number of crossings in any geometric graph with n vertices and $e > 4n$ edges is at least Ce^3/n^2, for a suitable constant $C > 0$. Thus, if Lovász' geometric graph G had more than $\gamma n^{3/2}$ edges, then it would have an edge crossing at least $2Ce^2/n^2 > 2C\gamma^2 n > n$ others, provided that $\gamma \geq 1/\sqrt{2C}$. This would contradict the property that every line crosses at most n edges of G. Using a more delicate argument based on the same idea, Dey [**D98**] showed that the number of halving lines is $O(n^{4/3})$. It is quite likely that this bound can be further improved.

The above examples suggest how to reformulate one of the fundamental questions of *extremal graph theory* for geometric graphs: What is the maximum number of edges that a geometric graph of n vertices can have if it satisfies certain geometric properties? The papers of Pach-Radoičić-Tóth, Székely, and Pinchasi-Radočić focus on problems of this type. The last paper represents a real breakthrough: it is shown that the number of edges of a geometric graph with n vertices that contains no self-intersecting cycle of length 4 is $O(n^{8/5})$. This result has already found

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several applications. Brass generalizes some extremal problems to geometric hypergraphs (e.g., to systems of triangles induced by a point set in convex position). The contributions of Cairns-McIntyre-Nikolayevsky and Perles-Pinchasi are related to Conway's Thrackle conjecture and to the problem of halving lines, respectively. Lovász-Vesztergombi-Wagner-Welzl discover a beautiful connection between k-sets and crossing numbers. As a consequence, they get surprisingly close to determining the smallest number of crossings that a complete geometric graph with n vertices can have. The papers of Eppstein and Shahrokhi-Sýkora-Székely-Vrťo analyze the relationship between other natural graph parameters related to edge crossings.

After the pioneering work of Székely [**Sz97**] and Solymosi-Tóth [**ST01**], there have been many important developments related to Erdős's question about the minimum number of distinct distances determined by n points in the plane [**E46**]. The paper of Katz-Tardos included in this volume represents the state of the art in this area: the best lower bound presently known for this function. On the other hand, Ruzsa's contribution indicates the theoretical limits of their approach. Solymosi-Vu make a big step towards an asymptotic solution of the analogous question in higher dimensions. Here they detail their elegant arguments only for "homogeneous" sets, but it is not hard to extend their proof to the general case. A construction of Swanepoel-Valtr improves the best known lower bound on the number of times the same distance can occur among n points on the sphere. The papers of Katona-Mayer-Woyczynski and Jamison address related questions for distance and slope distributions, respectively.

The basic problems of the *theory of random graphs* and *Ramsey theory* can also be rephrased for geometric graphs. Spencer's paper describes the expected behavior of the biplanar crossing number of random graphs, while the works of Dumitrescu-Radoičić, Kaneko-Kano-Suzuki, Kostochka, and Nešetřil-Solymosi-Valtr address Ramsey-type questions. The last paper proves the following remarkable result that can be regarded as a far-reaching generalization of Fáry's theorem that states that every planar graph permits a straight-line drawing: For any coloring with a finite number of colors of all segments induced by point pairs in the plane, there is a color such that every planar graph has a straight-line drawing in which all edges have that color. The papers of Alt-Knauer-Rote-Whitesides and Maehara are concerned with the mobility and rigidity of geometric graphs (linkages), respectively. In the work of Arutyunyants-Iosevich, some deep and surprising connections are uncovered between distance geometry and measure theory. Roughly speaking, they prove that Falconer's celebrated conjecture [**F85**], claiming that every d-dimensional set of Hausdorff dimension at least $d/2$ has positive Lebesgue measure, is almost surely true for random metrics. The paper of Dujmović-Wood applies Lovász' Local Lemma to construct almost optimally compact crossing-free embeddings of geometric graphs in 3-space with the property that all vertices are mapped to integer points.

Our collection is dedicated to two outstanding mathematicians and lifelong friends: Paul Turán and Paul Erdős. Turán's Brick Factory Problem was the first genuine optimization problem in graph drawing. The systematic study of extremal problems for geometric graphs was initiated many years ago by P. Erdős, M. Perles, and his student, Y. Kupitz [**K79**], and by S. Avital and H. Hanani [**AH66**].

The present volume is *neither* a conference proceedings (i.e., quasi-random collection of articles), *nor* a systematic review of the field. It is a careful selection of 25 invited and thoroughly refereed papers, reporting about exciting recent discoveries on the way "Towards a Theory of Geometric Graphs."

János Pach
Berkeley, October 2003

References

[ACN82] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi: Crossing-free subgraphs, *Annals of Discrete Mathematics* **12** (1982) 9–12.

[AH66] S. Avital and H. Hanani: Graphs (in Hebrew), *Gilyonot Lematematika* **3** (1966) 2–8.

[D98] T. K. Dey: Improved bounds on planar k-sets and related problems, *Discrete and Computational Geometry* **19** (1998), 373–382.

[E46] P. Erdős: On sets of distances of n points, *American Mathematical Monthly* 53 (1946), 248–250.

[F85] K. J. Falconer: On the Hausdorff dimension of distance sets, *Mathematika* **32** (1985), 206–212.

[GO04] J. E. Goodman and J. O'Rourke: *Handbook of Discrete and Computational Geometry, 2nd ed.*, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 2004, to appear.

[KST54] T. Kővári, V. T. Sós, and P. Turán: On a problem of K. Zarankiewicz, *Colloqium Mathematicum* **3**, 50–57.

[K79] Y. Kupitz: *Extremal Problems in Combinatorial Geometry, Aarhus University Lecture Notes Series* **53**, Aarhus University, Denmark, 1979.

[L71] L. Lovász: On the number of halving lines, *Annales Universitatis Scientarium Budapest, Eötvös, Sectio Mathematica* **14** (1971), 107–108.

[L83] T. Leighton: *Complexity Issues in VLSI, Foundations of Computing Series*, MIT Press, Cambridge, MA, 1983.

[ST01] J. Solymosi and Cs. Tóth: Distinct distances in the plane, *Discrete and Computational Geometry* **25** (2001), 629–634.

[SSzT84] J. Spencer, E. Szemerédi, and W. T. Trotter: Unit distances in the Euclidean plane, in: *Graph Theory and Combinatorics* (B. Bollobás, ed.), Academic Press, New York, 1984, 293–303.

[Sz97] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* 6 (1997), 353–358.

[T77] P. Turán: A note of welcome, *J. Graph Theory* **1** (1977), 7–9.

[W69] D. R. Woodall: Thrackles and deadlock, in: *Combinatorial Mathematics and Its Applications* (D.J.A. Welsh, ed.), Academic Press, London, 335–348.

Contents

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On the Complexity of the Linkage Reconfiguration Problem

Helmut Alt, Christian Knauer, Günter Rote, and Sue Whitesides

ABSTRACT. We consider the problem of reconfiguring a linkage of rigid straight segments from a given start to a given target position with a continuous nonintersecting motion. The problem is nontrivial even for trees in two dimensions since it is known that not all configurations can be reconfigured to a straight position. We show that deciding reconfigurability for trees in two dimensions and for chains in three dimensions is PSPACE-complete.

Section 1. Introduction

A *linkage* in d-space is a crossing-free straight line embedding of a graph in $\mathbb{R}^d$ where the edges are considered as rigid bars and the vertices are considered as hinges. A *reconfiguration* is a continuous motion of the vertices that preserves the lengths of the edges and never causes edges to collide.

We investigate several complexity issues concerning the reconfiguration of simple (non-crossing) polygonal chains of fixed-length segments in 3D and of simple trees of fixed-length segments in 2D. To be more precise, we consider the complexity of variants of the following

Linkage reconfiguration problem: Given two linkages S and T in d-space, can we reconfigure S into T?

In 2D, every simple polygonal chain of fixed length segments can be continuously moved to a straight configuration [**17, 7**]. Nevertheless, the general question is nontrivial since it is known that simple trees of fixed length segments cannot always be moved to configurations that are essentially flat, [**7**], and that there are polygonal chains in 3D that cannot be moved to a straightened configuration [**5, 2**].

In Section 2 we observe that the linkage reconfiguration problem is decidable in polynomial space for any fixed dimension d. In Section 3 we complement these upper bounds by showing that the reconfiguration problem for trees in 2D is PSPACE-hard. The technique we use is based on work of Joseph and Plantinga [**11**]. Finally,

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Key words and phrases. Computational Geometry, Linkage reconfiguration, PSPACE-completeness.

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in Section 4 we sketch an extension of the PSPACE-hardness result to the reconfiguration problem for chains moving in 3D.

Section 2. Deciding the reconfiguration problem in polynomial space

Any embedding of a linkage consisting of n links in d-dimensional space can be described by giving the dn coordinates of the vertices. Thus, any embedding can be represented as a point in $\mathbb{R}^{dn}$. Furthermore, the condition that no two links may intersect can be written as a formula in the first order theory of the reals, i.e., a formula that is built by joining atomic formulas which involve the variables that represent the coordinates of the vertices, using addition and multiplication as operators, and $\leq$ as a predicate, with the usual boolean connectives. Furthermore, for each link, we can add to this formula the condition that its two endpoints lie a fixed distance (the length of the link) apart.

The region where this formula is true is a semialgebraic subset $\mathcal{F}$ of $\mathbb{R}^{dn}$ of the *feasible* embeddings of the linkage. Assume that two linkages S and T are given. Obviously linkage S can be reconfigured into linkage T precisely when there exists a path within $\mathcal{F}$ starting at S and ending at T.

The application of this technique to motion planning problems was investigated first by Schwartz and Sharir [**15**] and shown to be decidable using the techniques of Collins [**6**], which were improved later by Canny [**3**]. Afterwards, Renegar [**14**] showed that problems of this kind are in PSPACE (also claimed by Canny [**4**]). So we can conclude

THEOREM 1. *The reconfiguration problem for arbitrary linkages in d-dimensional space, $d \in \mathbb{N}$, is in PSPACE.*

Lower bounds for the computational complexity of motion planning problems were first investigated by Reif [**12**], who sketched a PSPACE hardness result for a reconfiguration problem in which the moving object has an arbitrary number of degrees of freedom (the result was further elaborated in [**16**], pages 267-281.) The moving object consists of n polyhedra linked together at revolute joints. It moves in a 3D environment that contains polyhedral obstacles, and is to move from one given configuration to another. Hopcroft, Joseph, and Whitesides [**10**] proved the PSPACE-hardness of a reachability problem for a linkage moving in the plane. The linkage consists of rigid rods joined together at their endpoints, about which they may turn. Rods are allowed to cross over one another, and some endpoints are fixed to the plane. There are no obstacles in the environment. Joseph and Plantinga [**11**] proved a PSPACE-hardness result for a planar linkage consisting of a chain of links connected sequentially and moving in a polygonal environment.

There are many lower bound results for motion planning problems, but only a few of these give PSPACE hardness results. With the exception of [**11**], these PSPACE hardness results are not for chains of links, but rather, concern other kinds of objects or sets of objects. Two early examples are the following. Hopcroft, Schwartz, and Sharir [**9**] proved the PSPACE-hardness of a reconfiguration problem for sliding blocks in a polygonal environment in 2D, and Reif and Sharir [**13**] gave a PSPACE hardness result for an object moving in 3D among n moving obstacles.

Early results on algorithmic motion planning, including lower bound results, were reprinted in a volume edited by Schwartz, Sharir, and Hopcroft [**16**]. This volume includes [**15**], a detailed variation of [**12**], and [**10**].

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