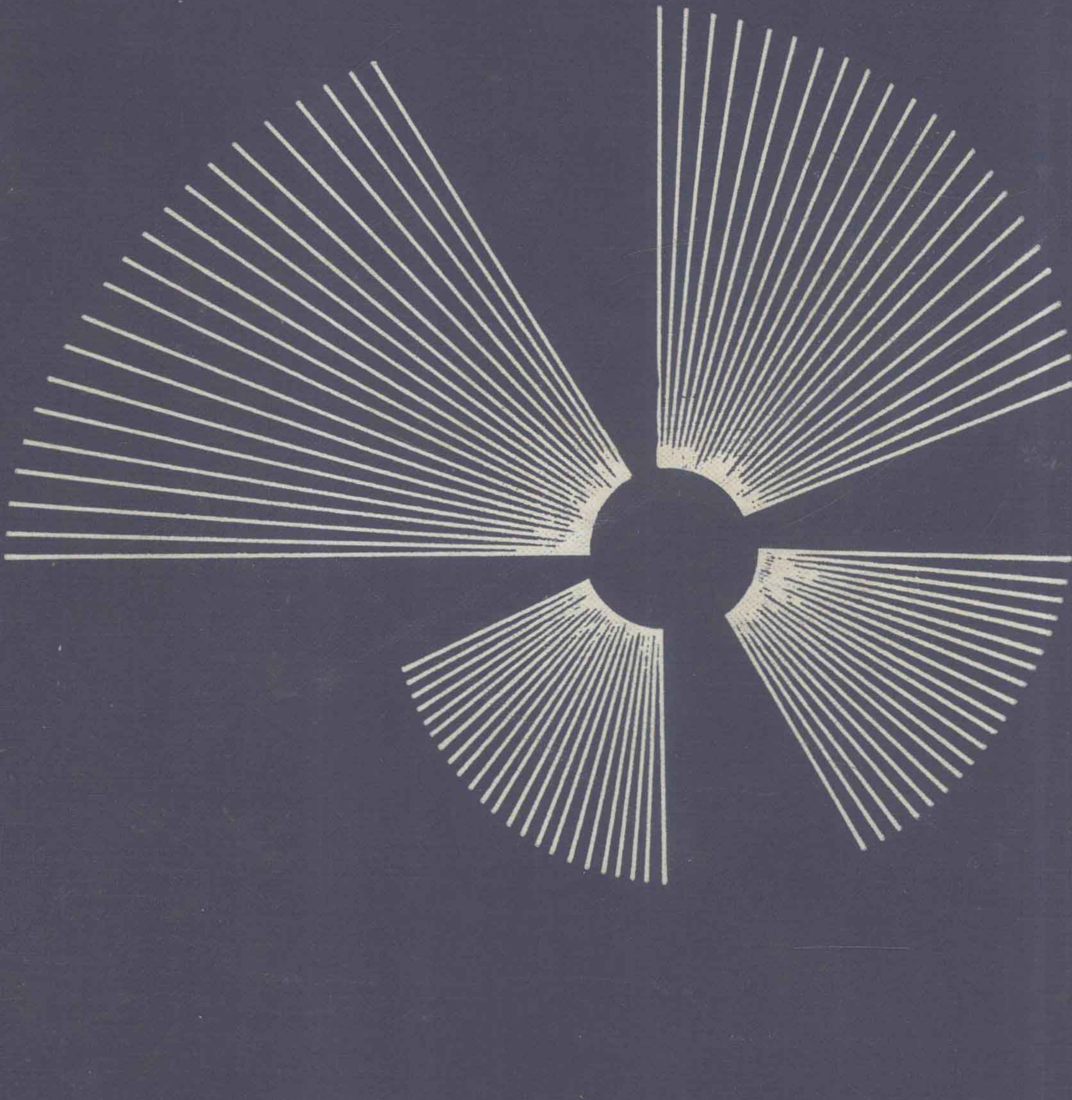

C. L. JOHNSTON

# PLANE TRIGONOMETRY

## A NEW APPROACH

# PLANE TRIGONOMETRY

## a new approach

**C. L. Johnston**

**East Los Angeles College**

APPLETON-CENTURY-CROFTS
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## SYMBOLS USED IN THIS COURSE

$<$ read *is less than*

$\leqq$ read *is less than or equal to*

$>$ read *is greater than*

$\geqq$ read *is greater than or equal to*

$\neq$ read *is not equal to*

$\approx$ read *is approximately equal to*

$\angle$ read *angle*

## GREEK LETTERS USED IN THIS COURSE

| | |
|---|---|
| $\alpha$ | Alpha |
| $\beta$ | Beta |
| $\gamma$ | Gamma |
| $\theta$ | Theta |
| $\pi$ | Pi |
| $\phi$ | Phi |

## IMPORTANT TRIGONOMETRIC FORMULAS

### Polar Coordinates

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$

$$y = r\sin\theta, \qquad \theta = \arctan\frac{y}{x}.$$

### The Area of a Sector

$$K = \tfrac{1}{2}r^2\theta.$$

### The Area of a Segment

$$K = \tfrac{1}{2}r^2(\theta - \sin\theta).$$

**Law of Sines**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

**Law of Cosines**

$$a^2 = b^2 + c^2 - 2bc\cos\theta,$$
$$b^2 = c^2 + a^2 - 2ca\cos\theta,$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

**Law of Tangents**

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}, \qquad \frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)},$$
$$\frac{c-a}{c+a} = \frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)}.$$

**The Half-angle Formulas in Terms of the Sides of a Triangle**

$$\tan\tfrac{1}{2}A = \frac{r}{s-a}, \qquad \tan\tfrac{1}{2}B = \frac{r}{s-b}, \qquad \tan\tfrac{1}{2}C = \frac{r}{s-c},$$
$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \sin\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$
$$\sin\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}},$$

where $\quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$ and $s = \frac{1}{2}(a+b+c).$

**The Area of a Triangle**

$$K = \tfrac{1}{2}ab\sin C.$$
$$K = \tfrac{1}{2}a^2\frac{\sin B\sin C}{\sin A}.$$
$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}.$

# PLANE TRIGONOMETRY

## a new approach

The Appleton-Century Mathematics Series

Raymond W. Brink and John M. H. Olmsted, Editors

# PREFACE

Students of Plane Trigonometry often encounter more difficulty in the study of the subject than is justified by the essential simplicity of the material. It is for this reason that in this book we avoid what appears to be one of the principal sources of their confusion.

In traditional textbooks on trigonometry, the student is introduced to all of the six basic trigonometric functions each immediately following the other. Then, very soon, he is required to give from memory, or to find from tables, the value of each of the six functions for an angle of any magnitude. For us who have lived with this subject for a long time, it requires no effort to determine both the correct algebraic sign and the numerical value, regardless of the value of the angle or real number on which the function depends. But experience shows that this is not such an easy matter for many students.

One of the principal innovations of this textbook is that here we take just one trigonometric function at a time and carry it through evaluations, trigonometric equations, applications to the right and the general triangles, and graphs. For example, by the time the student has lived with only the sine function through these applications, it has become thoroughly established in his mind and will not easily be forgotten. This is done with the sine function in Chapter 2, the cosine function in Chapter 3, and the tangent function in Chapter 4. Then, having this thorough knowledge of the sine, cosine, and tangent functions, the student has no difficulty extending his knowledge to include the definitions and applications of the reciprocal functions, which are treated in Chapter 5.

After studying the first five chapters a student should be able to (1) solve both right and general triangles and many basic problems involving vectors, (2) draw the graphs of all the trigonometric functions as well as composite functions, and (3) solve many trigonometric equations. In fact, for many students who are enrolled in some two-year curriculum such as drafting, mechanical engineering, or electronics, or some four-year curriculum such as medicine, dentistry, pharmacy, optometry, the first five chapters will give the necessary trigonometry. This does not mean that this textbook presents a "watered down" version of trigonometry. In fact I have used these first five chapters, experimentally, in classes for two semesters and found that one can ultimately cover the subject matter in less time by using this approach than can be done by the traditional method. This leaves more time for the subjects of the later chapters, such as inverse functions and complex numbers, which are often slighted at the end of the course. The entire book offers a complete course in Plane Trigonometry and is especially adaptable to courses of various lengths and purposes.

Modern notation has not been used, since many of our students do not have a background in modern mathematics and the use of modern notation would have required an additional introductory chapter and thus increased the length of the course. Since only a limited number of days are available for the course, the addition of new topics would leave less time for trigonometry. It is my opinion that to do a good job of teaching a complete course in plane trigonometry we must apply all of our time to that subject.

The unit circle has been used extensively as an aid in defining functions and developing formulas and identities.

An abundance of illustrative examples with solutions, two hundred seventy-five in number, anticipates the difficulties of the student and at the same time sets before him applications of basic principles and orderly solutions of exercises. The discussion of trigonometric graphs is more complete than in many other books on the subject. One hundred thirty-seven illustrative figures help to clarify the proofs, definitions, and examples. The thirteen hundred eighty-eight exercises allow a student to obtain practice on all parts of the theory by working either the odd-numbered or the even-numbered exercises. Answers to odd-numbered exercises appear at the end of the text. Answers to the even-numbered exercises are available in a separate pamphlet.

Basic trigonometric formulas are listed inside the front cover and the basic trigonometric identities inside the back cover so as to be immediately available when needed.

I am deeply indebted to Professor Raymond W. Brink, Consulting Editor of the Appleton-Century Mathematics Series, for his many suggestions and painstaking attention to my manuscript. The elegant proof of the Law of Cosines, the complete set of Tables, and more than a score of other passages in the text of this book were taken by permission from the *Third Edition* of Dr. Brink's *Plane Trigonometry*, Appleton-Century-Crofts, N.Y., 1959.

Whittier, California

C.L.J.

## TRIGONOMETRIC IDENTITIES

**(1)** $\csc\theta = \dfrac{1}{\sin\theta}$. **(2)** $\sec\theta = \dfrac{1}{\cos\theta}$. **(3)** $\cot\theta = \dfrac{1}{\tan\theta}$.

**(4)** $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. **(5)** $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

**(6)** $\sin^2\theta + \cos^2\theta = 1$. **(7)** $1 + \tan^2\theta = \sec^2\theta$. **(8)** $1 + \cot^2\theta = \csc^2\theta$.

**(9)** $\sin(-\theta) = -\sin\theta$. **(10)** $\cos(-\theta) = \cos\theta$. **(11)** $\tan(-\theta) = -\tan\theta$.

**(12)** $\cot(-\theta) = -\cot\theta$. **(13)** $\sec(-\theta) = \sec\theta$. **(14)** $\csc(-\theta) = -\csc\theta$.

**(15)** $\cos(A + B) = \cos A\cos B - \sin A\sin B$.

**(16)** $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

**(17)** $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

**(18)** $\sin(A - B) = \sin A\cos B - \cos A\sin B$.

**(19)** $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$.

**(20)** $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$.

**(21)** $\sin 2A = 2\sin A\cos A$.

**(22a)** $\cos 2A = \cos^2 A - \sin^2 A$.

**(22b)** $= 1 - 2\sin^2 A$.

**(22c)** $= 2\cos^2 A - 1$.

**(23)** $$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

**(24)** $$\sin \tfrac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{2}}.$$

**(25)** $$\cos \tfrac{1}{2}A = \pm\sqrt{\frac{1+\cos A}{2}}.$$

**(26)** $$\tan \tfrac{1}{2}A = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \text{unless } \cos A = -1.$$

**(27)** $$\sin A\cos B = \tfrac{1}{2}[\sin(A+B)+\sin(A-B)].$$

**(28)** $$\cos A\sin B = \tfrac{1}{2}[\sin(A+B)-\sin(A-B)].$$

**(29)** $$\sin A\sin B = \tfrac{1}{2}[\cos(A-B)-\cos(A+B)].$$

**(30)** $$\cos A\cos B = \tfrac{1}{2}[\cos(A+B)+\cos(A-B)].$$

**(31)** $$\sin x+\sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y).$$

**(32)** $$\sin x-\sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).$$

**(33)** $$\cos x+\cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y).$$

**(34)** $$\cos x-\cos y = -2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).$$

**(35)** $a\cos\theta + b\sin\theta = c\cos(\theta-\alpha)$,

where $c=\sqrt{a^2+b^2}$, $\sin\alpha = \dfrac{b}{c}$, $\cos\alpha=\dfrac{a}{c}$, $\tan\alpha=\dfrac{a}{b}$.

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# 1

# Introduction

## 101 TRIGONOMETRY

Trigonometry—a branch of mathematics that deals with the relationships between the angles and sides of triangles and the theory of the periodic functions connected with them—is a basic tool used in the development of mathematics and many sciences such as physics, engineering, astronomy, and the like.

## 102 RECTANGULAR COORDINATES

Although it is assumed that the student has had experience with the rectangular coordinate system, a brief review of the subject is given.

Two perpendicular lines are drawn meeting at $O$ (Fig. 101). The point $O$ is called the **origin**, the line $OX$ the **x-axis**, and the line $OY$ the **y-axis**. A convenient unit of length is used to mark off distances to the right and left and up and down from the origin $O$. Distances to the right are taken as positive values of $x$ and distances to the left are taken as negative values of $x$. Positive values of $y$ are measured upward and the negative values of $y$ are measured downward.

The position of any point $P$ on the $xy$-plane is determined by a pair of numbers called the **coordinates** of the point. The distance of $P$ to the right or left of the origin is called the **abscissa** or **x-coordinate** of point $P$, and $y$, the vertical distance of $P$ from the $x$-axis, is the **ordinate** or **y-coordinate** of point $P$. Point $P$ is said to have the coordinates $(x, y)$ and may be referred to as *the point* $(x, y)$.

**Example 1.** Locate the point (2, 3).

Start at the origin and move two units to the right, then up three units (Fig. 101).

**Example 2.** Locate the point (0, −2).

Start at the origin, but, because the first number is zero, do not move right or left. The second number being negative directs us down two units (Fig. 101).

**Example 3.** Locate the point (−3, 2).

Start at the origin and move left three units, then up two units (Fig. 101).

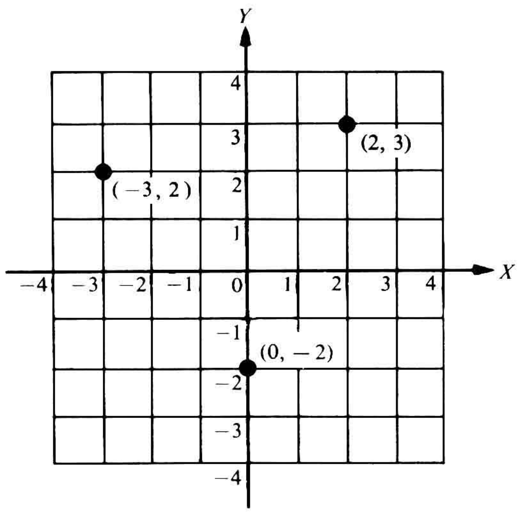


**Figure 101**

## 103 THE FORMATION OF ANGLES

A plane angle is formed if two half lines have the same end-point. This end-point is called the **vertex** of the angle and the two half lines are the **sides of the angle** (Fig. 102). We can think of the angle as being **generated** when a half line whose end-point is the vertex of the angle rotates in the plane about the vertex from the position of one side of the angle until it coincides with the