Partial Differential Equations

SECOND EDITION

Lawrence C. Evans

Graduate Studies in Mathematics

Volume 19

American Mathematical Society

This is the second edition of the now definitive text on partial differential equations (PDE). It offers a comprehensive survey of modern techniques in the theoretical study of PDE with particular emphasis on nonlinear equations. Its wide scope and clear exposition make it a great text for a graduate course in PDE. For this edition, the author has made numerous changes, including

- a new chapter on nonlinear wave equations,
- more than 80 new exercises,
- several new sections,
- a significantly expanded bibliography.

About the First Edition:

I have used this book for both regular PDE and topics courses. It has a wonderful combination of insight and technical detail. ... Evans' book is evidence of his mastering of the field and the clarity of presentation.

—Luis Caffarelli, University of Texas

It is fun to teach from Evans' book. It explains many of the essential ideas and techniques of partial differential equations ... Every graduate student in analysis should read it.

—David Jerison, MIT

I use Partial Differential Equations *to prepare my students for their Topic exam, which is a requirement before starting working on their dissertation. The book provides an excellent account of PDE's ... I am very happy with the preparation it provides my students.*

—Carlos Kenig, University of Chicago

Evans' book has already attained the status of a classic. It is a clear choice for students just learning the subject, as well as for experts who wish to broaden their knowledge ... An outstanding reference for many aspects of the field.

—Rafe Mazzeo, Stanford University

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Lawrence C. Evans

Department of Mathematics
University of California, Berkeley

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Partial Differential Equations

SECOND EDITION

I dedicate this book to the memory of my parents,

LAWRENCE S. EVANS and LOUISE J. EVANS.

Preface to the second edition

Let me thank everyone who over the past decade has provided me with suggestions and corrections for improving the first edition of this book. I am extraordinarily grateful. Although I have not always followed these many pieces of advice and criticism, I have thought carefully about them all. So many people have helped me out that it is unfortunately no longer feasible to list all their names. I have also received extraordinary help from everyone at the AMS, especially Sergei Gelfand, Stephen Moye and Arlene O'Sean. The NSF has generously supported my research during the writing of both the original edition of the book and this revision. I will continue to maintain lists of errors on my homepage, accessible through the math.berkeley.edu website.

When you write a big book on a big subject, the temptation is to include everything. A critic famously once imagined Tolstoy during the writing of *War and Peace*: "The book is long, but even if it were twice as long, if it were three times as long, there would always be scenes that have been omitted, and these Tolstoy, waking up in the middle of the night, must have regretted. There must have been a night when it occurred to him that he had not included a yacht race..." (G. Moore, *Avowals*).

This image notwithstanding, I have tried to pack into this second edition as many fascinating new topics in partial differential equations (PDE) as I could manage, most notably in the new Chapter 12 on nonlinear wave equations. There are new sections on Noether's Theorem and on local minimizers in the calculus of variations, on the Radon transform, on Turing instabilities for reaction-diffusion systems, etc. I have rewritten and expanded the previous discussions on blow-up of solutions, on group and phase velocities, and on several further subjects. I have also updated and greatly increased citations to books in the bibliography and have moved references to research articles to within the text. There are countless further minor modifications in notation and wording. Most importantly, I have added about 80 new exercises, most quite interesting and some rather elaborate. There are now over 200 in total.

And there is a yacht race among the problems for Chapter 10.

LCE
January, 2010
Berkeley

Preface to the first edition

I present in this book a wide-ranging survey of many important topics in the theory of partial differential equations (PDE), with particular emphasis on various modern approaches. I have made a huge number of editorial decisions about what to keep and what to toss out, and can only claim that this selection seems to me about right. I of course include the usual formulas for solutions of the usual linear PDE, but also devote large amounts of exposition to energy methods within Sobolev spaces, to the calculus of variations, to conservation laws, etc.

My general working principles in the writing have been these:

a. PDE theory is (mostly) not restricted to two independent variables. Many texts describe PDE as if functions of the two variables (x, y) or (x, t) were all that matter. This emphasis seems to me misleading, as modern discoveries concerning many types of equations, both linear and nonlinear, have allowed for the rigorous treatment of these in any number of dimensions. I also find it unsatisfactory to "classify" partial differential equations: this is possible in two variables, but creates the false impression that there is some kind of general and useful classification scheme available in general.

b. Many interesting equations are nonlinear. My view is that overall we know too much about linear PDE and too little about nonlinear PDE. I have accordingly introduced nonlinear concepts early in the text and have tried hard to emphasize everywhere nonlinear analogues of the linear theory.

c. Understanding generalized solutions is fundamental. Many of the partial differential equations we study, especially nonlinear first-order equations, do not in general possess smooth solutions. It is therefore essential to devise some kind of proper notion of generalized or weak solution. This is an important but subtle undertaking, and much of the hardest material in this book concerns the uniqueness of appropriately defined weak solutions.

d. PDE theory is not a branch of functional analysis. Whereas certain classes of equations can profitably be viewed as generating abstract operators between Banach spaces, the insistence on an overly abstract viewpoint, and consequent ignoring of deep calculus and measure theoretic estimates, is ultimately limiting.

e. Notation is a nightmare. I have really tried to introduce consistent notation, which works for all the important classes of equations studied. This attempt is sometimes at variance with notational conventions within a given subarea.

f. Good theory is (almost) as useful as exact formulas. I incorporate this principle into the overall organization of the text, which is subdivided into three parts, roughly mimicking the historical development of PDE theory itself. Part I concerns the search for explicit formulas for solutions, and Part II the abandoning of this quest in favor of general theory asserting the existence and other properties of solutions for linear equations. Part III is the mostly modern endeavor of fashioning general theory for important classes of nonlinear PDE.

Let me also explicitly comment here that I intend the development within each section to be rigorous and complete (exceptions being the frankly heuristic treatment of asymptotics in §4.5 and an occasional reference to a research paper). This means that even locally within each chapter the topics do not necessarily progress logically from "easy" to "hard" concepts. There are many difficult proofs and computations early on, but as compensation many easier ideas later. The student should certainly omit on first reading some of the more arcane proofs.

I wish next to emphasize that this is a *textbook*, and not a reference book. I have tried everywhere to present the essential ideas in the clearest possible settings, and therefore have almost never established sharp versions of any of the theorems. Research articles and advanced monographs, many of them listed in the Bibliography, provide such precision and generality. My goal has rather been to explain, as best I can, the many fundamental ideas of the subject within fairly simple contexts.

I have greatly profited from the comments and thoughtful suggestions of many of my colleagues, friends and students, in particular: S. Antman, J. Bang, X. Chen, A. Chorin, M. Christ, J. Cima, P. Colella, J. Cooper, M. Crandall, B. Driver, M. Feldman, M. Fitzpatrick, R. Gariepy, J. Goldstein, D. Gomes, O. Hald, W. Han, W. Hrusa, T. Ilmanen, I. Ishii, I. Israel, R. Jerrard, C. Jones, B. Kawohl, S. Koike, J. Lewis, T.-P. Liu, H. Lopes, J. McLaughlin, K. Miller, J. Morford, J. Neu, M. Portilheiro, J. Ralston, F. Rezakhanlou, W. Schlag, D. Serre, P. Souganidis, J. Strain, W. Strauss, M. Struwe, R. Temam, B. Tvedt, J.-L. Vazquez, M. Weinstein, P. Wolfe, and Y. Zheng.

I especially thank Tai-Ping Liu for many years ago writing out for me the first draft of what is now Chapter 11.

I am extremely grateful for the suggestions and lists of mistakes from earlier drafts of this book sent to me by many readers, and I encourage others to send me their comments, at evans@math.berkeley.edu. I have come to realize that I must be more than slightly mad to try to write a book of this length and complexity, but I am not yet crazy enough to think that I have made no mistakes. **I will therefore maintain a listing of errors which come to light, and will make this accessible through the math.berkeley.edu homepage.**

Faye Yeager at UC Berkeley has done a really magnificent job typing and updating these notes, and Jaya Nagendra heroically typed an earlier version at the University of Maryland. My deepest thanks to both.

I have been supported by the NSF during much of the writing, most recently under grant DMS-9424342.

LCE

August, 1997

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