Multivariate Spline Functions and Their Applications

Renhong Wang

SCIENCE PRESS
Beijing

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This book deals with the algebraic geometric method of studying multivariate splines. Topics treated include: the theory of multivariate spline spaces, higher dimensional spline, rational spline, piecewise algebraic variety (including piecewise algebraic curve and surface) and applications in the finite element method and computer aided geometric design. Many new results are given.

This volume will be of interest to researchers and graduate students whose work involves approximations and expansions, numerical analysis, computational geometry, image processing and CAD/CAM.

ISBN 7-03-007897-7

9 787030 078971 01>

ISBN 7-03-007897-7
定 价： 120.00 元

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Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China
Second Printed in 2006

Printed in Beijing

ISBN 7-03-007897-7 (Science Press, Beijing)

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Preface

As is known, the book named "Multivariate spline functions and their applications" has been published by the Science Press in 1994.

This book is an English edition based on the original book mentioned above with many changes, including that of the structure of a cubic C^1-interpolation in n-dimensional spline spaces, and more detail on triangulations have been added in this book.

Special cases of multivariate spline functions (such as step functions, polygonal functions, and piecewise polynomials) have been examined mathematically for a long time. I. J. Schoenberg (*Contribution to the problem of application of equidistant data by analytic functions, Quart. Appl. Math.*, $4(1946), 45-99; 112-141$) and W. Quade & L. Collatz (*Zur Interpolations theories der reellen periodischen function, Press. Akad. Wiss. (PhysMath. KL)*, $30(1938), 383-429$) systematically established the theory of the spline functions. W. Quade & L. Collatz mainly discussed the periodic functions, while I. J. Schoenberg's work was systematic and complete. I. J. Schoenberg outlined three viewpoints for studing univariate splines: Fourier transformations, truncated polynomials and Taylor expansions. Based on the first two viewpoints, I. J. Schoenberg deduced the B-spline function and its basic properties, especially the basis functions. Based on the latter viewpoint, he represented the spline functions in terms of truncated polynomials. These viewpoints and methods had significantly effected on the development of the spline functions.

In view of the variety and complexity in application, it is very important to study the multivariate spline function theoretically. Since the multivariate spline function is heavily dependent on the geometric property of the domain partitions, it is so complex that the multivariate spline function, especially the non-Cartesian product multivariate spline function, has not been developed radically for a long time. G. Birkhoff, H. L. Garabedian, C. de Boor, M. H. Schultz and R. S. Varga discussed the Çartesian product bicubic spline function and its applications in numerical solutions of partial differential equations.

Analysing the relation between the polynomials over two adjacent cells, we introduce the smooth cofactor and conformality condition to which the polynomials must satisfy. The conformality condition establishes the equivalent conversion between the multivariate spline function and the corresponding algebraic problem. Moreover, the conformality condition provides an algebraic approach to studying the multivariate spline function. Based on the conformality condition theory, we have systematically studied the dimension of the multivariate spline functions, the basis functions, especially the locally supported basis functions, the smooth surface interpolations, the non-linear spline interpolations, the higher-dimensional spline functions, and the multivariate spline functions in computer aided geometric designs.

This book will systematically introduce the basic theories and methods on the multivariate spline functions. In order for the reader to know the frontier research on the multivariate spline functions, we will also introduce the modern developments of the multivariate spline functions and their applications in sciences and engineering. More precisely, Chapter 1 introduces the basic definitions of the multivariate spline functions, facts, and results; Chapter 2 mainly introduces the dimension of the multivariate spline function space the theory on the basis functions, and their constructions; Chapter 3 mainly introduces the notable Box spline, the simplex spline, and the B-net method, etc.; Chapter 4 introduces the basic theory, methods, and structures of the higher-dimensional spline functions; Chapter 5 introduces the theory on non-linear spline interpolations and their constructive methods; Chapter 6 introduces the basic problems and results on the piecewise algebraic curves and the piecewise algebraic surfaces; Chapter 7 introduces applications of the multivariate spline functions in the sciences and engineering, especially in finite element methods and computer aided geometric designs.

The writing of this book was participated in by professors Xiquan Shi, Zhongxuan Luo, Zhixun Su, and Dr. Shao-Ming Wang who is also the translator of this book. I wish to express my great appreciation to the Publishing Foundation of Academia Sinica, as will as The National

Nature Science Foundation of China. Without their assistance, this book is unable to be published.

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October, 2000

Contents

Chapter 1

Introduction to Multivariate Spline Functions

It is well known that spline functions play very important roles in both theories and applications in the sciences and engineering. In view of the variety and complexity of the objectives, it is important to study the multivariate splines. Between the 1960's and early the 1970's, G. Birkhoff, H. L. Garabedian and Carl de Boor studied and established a series of theories on Cartesian tensor product multivariate splines. Although the Cartesian tensor product multivariate spline has its own application value, they are a simple extension of univariate spline functions, so they have many limitations.

In 1975, the author established a new approach to studying the basic theory of multivariate spline functions using the methods of function theory and algebraic geometry, and presented the so-called of smooth cofactor conformality method. Making use of this method, any problem on multivariate spline functions can be studied by transferring it into an equivalent algebraic problem.

Let D be a two dimensional domain in R^2, $\mathbf{P}_k$ be the collection of all these bivariate polynomials with real coefficients and total degree $\leq k$:

$$\mathbf{P}_k := \{\, p = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij} x^i y^j \mid c_{ij} \text{ is a real value}\}.$$

A bivariate polynomial $p \in \mathbf{P}_k$ is called an irreducible polynomial if the polynomial can not be exactly divided by any other polynomial except a constant or itself (in the complex field). An algebraic curve

$$\Gamma : l(x,y) = 0, \quad l(x,y) \in \mathbf{P}_m$$

is called an irreducible algebraic curve if $l(x,y)$ is an irreducible polynomial. Clearly, a straight line is an irreducible algebraic curve.

Using a finite number of irreducible algebraic curves to carry out the partition Δ in a domain D, the domain D is divided into a finite number of sub-domains $D_1, D_2, \cdots, D_N$ by the partition Δ; each of such sub-domains is called a 'cell'. These line segments that form the boundary of each cell are called the 'mesh segments' (edge); intersection points of the mesh segments are called the 'mesh points' (vertex). The interior of a mesh segment has no mesh point, that is, only the two ends of the mesh segments are mesh points. All mesh points in a closed cell are called the vertices of this cell. If two mesh points are two end points of a single mesh segment, then these two mesh points are called adjacent mesh points.

Carrying out the partition Δ in the domain D, the union of all the cells with a certain mesh point V as a vertex is called an incidence domain or a star shape domain of the mesh point V relative to the partition Δ, denoted by $St(V)$.

The space of multivariate spline functions is defined by

$$S_k^{\mu}(\Delta) := \{s \in C^{\mu}(D) \mid \Phi \mid_{D_i} \in \mathbf{P}_k, i = 1, \cdots, N\}.$$

In fact, $s \in S_k^{\mu}(\Delta)$ is a piecewise polynomial of degree k possesses μ order continuous partial derivatives in D.

1.1 Basic frame of multivariate spline functions

In order to establish the basic frame of multivariate spline functions, we need the following lemmas.

Lemma 1.1 [1] *Let $p(x,y) \in \mathbf{P}_k$. If certain n zeros $(x_i, y_i)(i = 1, \cdots, n)$, $n \geq k+1$ of a linear polynomial*

$$l(x,y) = ax + by + c, \qquad a^2 + b^2 \neq 0$$