An Introduction to Extremal Kähler Metrics

Gábor Székelyhidi

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To Sonja and Nóra

Preface

This book grew out of lecture notes written for a graduate topics course taught at the University of Notre Dame in the spring of 2012. The goal is to quickly introduce graduate students to ideas surrounding recent developments on extremal Kähler metrics. We make an effort to introduce the main ideas from Kähler geometry and analysis that are required, but the parts of the book on geometric invariant theory and K-stability would be difficult to follow without more background in complex algebraic geometry. A reader with a background in Riemannian geometry and graduate level analysis should be able to follow the rest of the book.

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Gábor Székelyhidi
Notre Dame, 2014

Introduction

A basic problem in differential geometry is to find canonical, or best, metrics on a given manifold. There are many different incarnations of this, perhaps the most well known being the classical uniformization theorem for Riemann surfaces. The study of extremal metrics is an attempt at finding a higher-dimensional generalization of this result in the setting of Kähler geometry. Extremal metrics were introduced by Calabi in the 1980s as an attempt to find canonical Kähler metrics on Kähler manifolds as critical points of a natural energy functional. The energy functional is simply the L^2-norm of the curvature of a metric. The most important examples of extremal metrics are Kähler-Einstein metrics and constant scalar curvature Kähler (or cscK) metrics.

It turns out that extremal metrics do not always exist, and the question of their existence is particularly interesting on projective manifolds. In this case, by works of Yau, Tian, and Donaldson, it was realized that the existence of extremal metrics is related to the stability of the manifold in an algebro-geometric sense, and obtaining a necessary and sufficient condition of this form for existence is the central problem in the field. Our goal in this book is to introduce the reader to some of the basic ideas on both the analytic and the algebraic sides of this problem. One concrete goal is to give a fairly complete proof of the following result.

Theorem. *If M admits a cscK metric in $c_1(L)$ for an ample line bundle $L \to M$ and if M has no non-trivial holomorphic vector fields, then the pair (M, L) is K-stable.*

The converse of this result, i.e. the existence of cscK metrics on K-stable manifolds, is the central conjecture in the field.

We will now give a brief description of the contents of the book. The first two chapters give a quick review of some of the background material that is needed. The first chapter contains the basic definitions in Kähler geometry, with a focus on calculations in local coordinates. The second chapter focuses on some of the analytic background required, in particular the Schauder estimates for elliptic operators, which we prove using a blow-up argument due to L. Simon.

The topic of Chapter 3 is Kähler-Einstein metrics, which are a special case of extremal metrics. We give a proof of Yau's celebrated theorem on the solution of the complex Monge-Ampère equation, leading to existence results for Kähler-Einstein metrics with zero or negative Ricci curvature. The case of positive Ricci curvature has only been understood very recently through the work of Chen, Donaldson, and Sun. The details of this are beyond the scope of this book, and we only give a very brief discussion in Section 3.5.

The study of general extremal metrics begins in Chapter 4. Following Calabi, we introduce extremal metrics as critical points of the Calabi functional, which is the L^2-norm of the scalar curvature:

$$\omega \mapsto \int_M S(\omega)^2 \, \omega^n,$$

defined for metrics ω in a fixed Kähler class. An important discovery is that extremal metrics have an alternative variational characterization, as critical points of the (modified) Mabuchi functional. This is convex along geodesics in the space of Kähler metrics with respect to a natural, infinite-dimensional, Riemannian structure. Moreover the variation of the Mabuchi functional is closely related to the Futaki invariant, which plays a prominent role in the definition of K-stability. After giving the basic definitions, we construct an explicit family of extremal metrics on a ruled surface due to Tønnesen-Friedman in Section 4.4. This example illustrates how a sequence of extremal metrics can degenerate, and we return to it again in Section 6.5. In Section 4.5 we give an introduction to the study of extremal metrics on toric manifolds. Toric manifolds provide a very useful setting in which to study extremal metrics and stability, and while in the two-dimensional case the basic existence question is understood through the works of Donaldson and of Chen, Li, and Sheng, the higher-dimensional case remains an important problem to study.

In Chapter 5 we give an introduction to the relation between symplectic and algebraic quotients—the Kempf-Ness theorem—which, at least on a heuristic level, underpins many of the ideas that have to do with extremal metrics. The general setting is a compact group K acting by Hamiltonian isometries on a Kähler manifold M, with a moment map $\mu : M \to \mathfrak{k}^*$. The

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Kempf-Ness theorem characterizes those orbits of the complexified group K^c which contain zeros of the moment map. The reason why this is relevant is that the scalar curvature of a Kähler metric, or rather the map $\omega \mapsto S(\omega) - \hat{S}$ where $\hat{S}$ is the average scalar curvature, can be realized as a moment map for a suitable infinite-dimensional Hamiltonian action. At the same time, orbits of K^c can be thought of as metrics in a given Kähler class, so an infinite-dimensional analog of the Kempf-Ness theorem would describe Kähler classes that contain cscK metrics. In Section 5.5 we will describe a suitable extension of the Kempf-Ness theorem dealing with critical points of the norm squared of a moment map, which in the infinite-dimensional setting are simply extremal metrics.

The notion of K-stability is studied in Chapter 6. It is defined in analogy with the Hilbert-Mumford criterion in geometric invariant theory by requiring that a certain weight—the Donaldson-Futaki invariant—is positive for all $\mathbf{C}^*$-equivariant degenerations of the manifold. These degenerations are called test-configurations. In analogy with the finite-dimensional setting of the Kempf-Ness theorem, the Donaldson-Futaki invariant of a test-configuration can be seen as an attempt at encoding the asymptotics of the Mabuchi functional "at infinity", with the positivity of the weights ensuring that the functional is proper. In Section 6.6 we will describe test-configurations from the point of view of filtrations of the homogeneous coordinate ring of the manifold. It is likely that the notion of K-stability needs to be strengthened to ensure the existence of a cscK metric, and filtrations allow for a natural way to enlarge the class of degenerations that we consider. In the case of toric varieties, passing from test-configurations to filtrations amounts to passing from rational piecewise linear convex functions to all continuous convex functions, as we will discuss in Section 6.7.

The basic tool in relating the differential geometric and algebraic aspects of the problem is the Bergman kernel, which we discuss in Chapter 7. We first give a proof of a simple version of the asymptotic expansion of the Bergman kernel going back to Tian, based on the idea of constructing peaked sections of a sufficiently high power of a positive line bundle. Then, following Donaldson, we use this to show that a projective manifold which admits a cscK metric must be K-semistable. This is a weaker statement than the theorem stated above. The Bergman kernel also plays a key role in the recent developments on Kähler-Einstein metrics, through the partial C^0-estimate conjectured by Tian. We will discuss this briefly in Section 7.6.

In the final chapter, Chapter 8, the main result is a perturbative existence result for cscK metrics due to Arezzo and Pacard. Starting with a cscK metric ω on M and assuming that M has no non-zero holomorphic vector fields, we show that the blow-up of M at any point admits cscK metrics in

suitable Kähler classes. The gluing technique used together with analysis in weighted Hölder spaces has many applications in geometric analysis. Apart from giving many new examples of cscK manifolds, this existence result is crucial in the final step of proving the theorem stated above, namely to improve the conclusion from K-semistability (obtained in Chapter 7) to K-stability. The idea due to Stoppa is to show that if M admits a cscK metric and is not K-stable, then a suitable blow-up of M is not even K-semistable. Since the blow-up admits a cscK metric, this is a contradiction.

There are several important topics that are missing from this book. We make almost no mention of parabolic equations such as the Calabi flow and the Kähler-Ricci flow. We also do not discuss in detail the existence theory for constant scalar curvature metrics on toric surfaces and for Kähler-Einstein metrics on Fano manifolds since each of these topics could take up an entire book. It is our hope that after studying this book the reader will be eager and ready to tackle these more advanced topics.

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