ZOFIA SZMYDT

FOURIER TRANSFORMATION AND LINEAR DIFFERENTIAL EQUATIONS

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Translated from the Polish by

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PREFACE

The present book has developed from lectures which I held at the Jagiellonian University (Cracow), at the Mathematical Institute of the Polish Academy of Science and at the Warsaw University during the years 1966–74. The first written form of those lectures was the book under the same title (in Polish) published by the Polish Scientific Publishers, Warsaw, in 1972. The present edition is an extended version of the Polish edition.

In the preparation of this book I have made heavy use of the monographs by L. Schwartz [47], L. Hörmander [29] and of the concepts introduced by S. Łojasiewicz in his papers [37] and [38].

The main purpose of this book is a presentation, in a distributional setting, of the simplest limit problems for the basic operators of mathematical physics. It has proved advisable to develop the technique of distribution-valued functions. In all problems considered one of the variables plays a special role. If the limit problem in question concerns the "upper" half-space (with respect to the distinguished variable), i.e., if that variable assumes positive values only, then the solution of the problem can be reduced to the construction of an appropriate distribution-valued function defined on the interval $(0, +\infty)$. Such a function is then constructed by a method consisting in an application of the Fourier transformation.

By a solution of a differential equation we mean any distribution satisfying this equation. Such a distribution can, of course, be in fact a function of a sufficient class of regularity; then it is a classical solution. If the data occurring in the limit problems considered are sufficiently regular functions, then every distributional solution is a classical one. Thus classical solutions are obtained from distributional solutions without resorting to the classical theory of partial differential equations. Consequently the text is intelligible to students unfamiliar with such equations and can serve as a manual which is to acquaint the reader with the basic topics of that domain and as an introduction into the modern theory of linear differential operators (cf. [29], [63]). Nevertheless, it should be stressed that the book is a monograph and not a course on differential equations: there are many important topics which it does not touch upon such as e.g. the eigenvalue problem or the theory of characteristics and the classification of differential equations. Course expositions of those topics can be found in [16], [61], [11], [33].

The book is self-contained. It presupposes only the knowledge of the elementary calculus (including ordinary differential equations and functions of one complex variable). A few theorems of functional analysis which are applied in the text have for convenience been formulated in the Introduction.

The first two chapters contain the theory of distributions and the Fourier transformation, and should be regarded as a basis for the sequel, dealing with partial differential operators. Besides the material contained in all text books on distributions, those chapters are dealing also with certain specific topics: the concept of distribution-valued functions and their properties, and the operation of fixing variables in a distribution (§§ 12, 13). To each of these notions corresponds its analogue concerning the case of tempered distributions (§ 19). Within this group of topics we might perhaps include also the operation of substitution in distributions (§§ 6.2 and 20.7),[1] emphasis being put on homogeneous distributions. Their properties are employed in Chapter VI and in the Appendix; it is also worth noticing that they play an important role in the theory of pseudo-differential operators, a theory developing very intensively.

The exposition of the theory of distributions follows the functional approach founded by Sobolev and Schwartz. It is preceded by a short introduction on vector spaces in which convergence has been defined by a sequence of semi-norms (§ 1).

Chapter III contains general definitions and theorems concerning linear differential equations and presents certain methods useful in solving such equations. Subsequent chapters are devoted to the study of three principal differential operators of mathematical physics: the wave operator (Chapter IV), the operator of heat conduction (Chapter V),

[1] The first number refers to section, the second to subsection. Thus § 6.2 means: Subsection 2 of Section 6.

and the Laplace and Helmholtz operators (Chapter VI). Besides Chapters I–VI the book contains an Appendix written in January 1976.

The Fourier transformation is a powerful tool, which allows one to obtain fundamental solutions of the above operators in a very natural way. A partial differential equation is transformed into an easily solvable ordinary equation, and it remains to "re-translate" the solution by applying the inverse Fourier transformation. This method is not necessarily the simplest one (the shortest way generally consists in a direct verification that the given distribution is indeed a fundamental solution); however, it has the advantage that it always leads to a solution and does not require its previous knowledge (or guessing). The difficulties which arise are rather of technical nature and consist in the computation of the Fourier transforms involved. Those transforms which are necessary in the application to the operators dealt with in Chapters IV–VI are computed in the examples of Chapter II (§ 21).

The first two chapters can also serve as a manual for an introductory course on distributions for mathematicians.[1] Then the monographs [47], [62], [30] and [21]–[23] can be suggested for further study. The exposition of the theory of distributions in Schwartz's monograph [47] follows the style and methods of the treatise of Bourbaki [6] and [7] and often resorts to it. Monographs [62] and [30] themselves contain a course on topological vector spaces. Other approaches to the theory of distributions can be found in [35] and [1].[2]

In numerous books distributions are treated with special regard to the application to differential equations. This concerns, in particular, the above-mentioned monographs [47] and [21]–[23], and also many others, e.g. [48], [65], [4], [10]. A brief survey of the basic properties of distributions and of the Fourier transformation with an application in the theory of holomorphic functions is given in [9]; applications in harmonic analysis are pointed out in [12].

Books [5], [13], [65] are texts for a course on differential equations; they are all based on the theory of distributions. In [13] emphasis is put

[1] Regarding those chapters as such, the reader can omit §§ 19 and 21 without loss.

[2] Some of the bibliographical items are mentioned only here, in the preface, just to make it clear that the theory of distributions admits various ways of approach and development, as well as diverse possibilities of applications.

on applications rather than theory. Moreover, various applications of distributions and differential operators in physics and technology can be found in [3], [48], [50]. The fundamental work which presents differential equations of mathematical physics in a classical setting is [11]. [33] is similar in character.

In the present book the material is exposed so as to be intelligible to beginners. Most of theorems are proved in detail. Exercises are numerous and are closely connected with the main text. Some of them are just illustrations, others are complement to the text. Sometimes they contain facts important for further reading and are referred to in subsequent sections. Occassionally also proofs of theorems omitted in the main text are placed among the exercises. Such a system has been adopted in order to encourage the reader to active reading. More difficult exercises are supplied with hints, which sometimes give, in fact, an outline of the proof. Those exercises which can be omitted without loss for further reading and are relatively difficult are marked with an asterisk.

The book consists of 44 sections (abbreviated: §§) numbered continuously throughout the Introduction and all the 6 chapters. Some sections are divided into subsections. Theorems, propositions, lemmas, corollaries, examples, formulae, exercises and also subsections are numbered independently in each section. Referring to a theorem (a proposition, etc.) within one section, we use only its successive number in that section; e.g., we write: see Theorem 3. Referring to a theorem (etc.) in other sections, we use double numbering, the first number referring to the section and the second indicating the successive item within it; thus Theorem 7.3 denotes Theorem 3 in § 7. A similar rule applies to subsections, as was mentioned in footnote 1 on p. VIII. Numbers in brackets denote formulae: e.g. (12) refers to the 12th formula in the same section, while (7.12) is the 12th formula in § 7. Numbers in square brackets refer to the bibliography.

Bibliographical references are mostly given in footnotes, which are numerous and contain various comments on the text. Footnotes are referred to by quoting the note number and the page.

The book is supplemented with indexes of names, subjects and symbols.

February 25, 1975 *Zofia Szmydt*

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TO THE ENGLISH EDITION

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Chapter VI. Laplace, Poisson and Helmholtz Equations

INTRODUCTION

§ 0. TERMINOLOGY AND NOTATION

We employ the usual notation of set theory. The union, the intersection and the difference of sets A and B are denoted by $A \cup B$, $A \cap B$, $A \setminus B$, respectively. $\emptyset$ is the empty set. We write $a \in A$ if a is an element of A, otherwise we write $a \notin A$. The notation $A \subset B$ (or $B \supset A$) means that A is a subset of B. If X is some space containing A, then the set $X \setminus A$ is called the *complement* of A (with respect to X) and is denoted by the symbol $\mathrm{C}A$.

The set of all elements of a set A satisfying condition $f(\cdot)$ is denoted by $\{a \in A: f(a)\}$, or shortly $\{a: f(a)\}$ if the set A is fixed and no confusion is likely to arise.

The *product* of sets A and B is the set $A \times B$ consisting of all ordered pairs (a, b) with $a \in A$, $b \in B$.

The notation

$$u\colon A \to B \quad \text{or} \quad A \ni a \mapsto u(a) \in B$$

means that u is a function whose domain is the set A and whose counterdomain is a subset of B. This is also read as: "u is a mapping of (the set) A into (the set) B", or "u is a function defined on (the set) A with values in (the set) B", or else "u is a B-valued function defined on A". The words "mapping" and "function" have the same meaning. For certain special classes of functions we shall use the expressions: operation, operator, transformation. A *functional* is a number-valued function. The supremum of a real-valued function on a set A is denoted by $\sup u$ or $\sup_A u$, or $\sup_{a \in A} u(a)$. Similarly the infimum of u on A is written as $\inf u$ or $\inf_A u$, or $\inf_{a \in A} u(a)$. Analogous notation is employed for $\max u$ and $\min u$ on A.

If for any $b \in B$ there is an $a \in A$ such that $u(a) = b$, we say that the function u *maps* the set A *onto* the set B. A function with this property is also called a *surjection* of A onto B. If u maps A into (but not necessarily onto) B in a *one-to-one* way, i.e., if we have $u(a_1) \neq u(a_2)$ for $a_1 \neq a_2$, then we say that u is an *injection* or an *embedding* of A into B. A mapping which is both an injection and a surjection is called a *bijection*. If $u: A \to B$ is a bijection, then its *inverse* $u^{-1}: B \to A$ is defined by: $u^{-1}(b) = a$ iff $u(a) = b$ (*iff* is an abbreviation for: if and only if).

If $u: A \to B$ is any mapping and if $\tilde{A}$ is a subset of A, then the *restriction* of u from A to $\tilde{A}$ is the mapping: $\tilde{A} \ni a \mapsto u(a) \in B$.

If $u: A \to B$, $v: B \to C$, then the *superposition* of u and v, i.e. the mapping $A \ni a \mapsto v\big(u(a)\big) \in C$, is denoted by $v \circ u$.

Suppose that v is a function defined in the product of two sets A, B and has values in a set C:

$$A \times B \ni (a, b) \mapsto v(a, b) \in C.$$

Then for any fixed values $\tilde{b} \in B$, $\tilde{a} \in A$ there are well defined mappings

$$A \ni a \mapsto v(a, \tilde{b}) \in C, \qquad B \ni b \mapsto v(\tilde{a}, b) \in C.$$

We denote these mappings by the symbols $v(\cdot, \tilde{b})$, $v(\tilde{a}, \cdot)$, respectively.

E denotes the set of real numbers. The *n-dimensional Euclidean space* is the product of n copies of the set E, i.e., the set of all points $x = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are finite real numbers. The n-dimensional Euclidean space is denoted by E^n. The number x_i $(i = 1, \ldots, n)$ is called the *i-th coordinate* of the point x. We denote by $|x|_2$ the *length* of the vector x (the *Euclidean norm* of the point x), i.e., the number given by

$$|x|_2^2 = x_1^2 + \ldots + x_n^2.$$

If $x = (x_1, \ldots, x_n) \in E^n$, $y = (y_1, \ldots, y_n) \in E^n$, we write

$$xy = x_1 y_1 + \ldots + x_n y_n$$

and call this number the *scalar product* of x and y.

For an $r > 0$, $x \in E^n$, the set

$$B(x, r) = \{y: |y-x|_2 < r\}$$

is called the *open ball* with centre x and radius r; for $0 < r_1 < r_2$, $x \in E^n$, the set

$$P(x, r_1, r_2) = \{y: r_1 < |y-x|_2 < r_2\}$$