EXERCISES IN MATHEMATICS

J. BASS

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Paris, France

Simple and multiple integrals. Series of functions.
Fourier series and Fourier integrals. Analytic functions.
Ordinary and partial differential equations.

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Preface

It is not without some hesitation that I have decided to publish this volume of solved mathematical problems. For while it is essential to offer the student a large choice of exercises, it is less evident that providing detailed models of solutions will be useful for him. More precisely, one might feel that a small number of such models, judiciously chosen, should normally form a part of the mathematical treatises themselves and that this should be sufficient. However, after several years of experience, I have come to feel that a collection of exercises, varied in nature and worked out in detail, can be beneficial, and that the student needs to learn how to work them out as much as he does to reason in connection with them and to find their solutions.

This collection is intended for students of applied mathematics, physics, and engineering. The spirit in which it is written corresponds approximately to that of the upper division level of instruction in courses on mathematical techniques. The subjects treated, however, are often found in courses on a somewhat lower level (e.g., integral calculus) as well as a somewhat higher level (e.g., mathematical methods in physics).

These exercises do not pretend in any way to take the place of a course of instruction. To help the reader, each chapter is preceded by a brief introduction in which certain essential formulas are recalled and in which the prerequisites are briefly outlined. Of course, such prerequisites should have already been acquired in oral or written instruction.

Although there are few books covering all the material, one can find the substance of it in quite a large number of works under the general headings of applied mathematics, advanced calculus, matrix and tensor calculus, etc., written for the upper undergraduate level and designed for engineers and physicists. Among the texts that are best suited we may cite:

R. Courant and D. Hilbert, "Methods of Mathematical Physics," Volume 1. Wiley (Interscience), New York, 1953.

This is a fundamental reference work which has served as a model for all advanced books in applied mathematics. One need not have studied it in detail to be able to do these exercises, but familiarity with certain chapters is essential.

R. V. Churchill, "Complex Variables and Applications," 2nd ed. McGraw-Hill, New York, 1960.

R. Courant and F. John, "Introduction to Calculus and Analysis." Wiley (Interscience), New York, 1965.

J. Irving and N. Mullineux, "Mathematics in Physics and Engineering." Academic Press, New York, 1959.

I. S. Sokolnikoff and R. M. Redheffer, "Mathematics of Physics and Modern Engineering." McGraw-Hill, New York, 1958.

It is difficult to state explicitly the sources of the exercises whose solutions are worked out in the present collection. Some of them are original. Many have been suggested in a more or less conscious way by analogous exercises that appear rather ubiquitously in mathematical literature. Some of them are of the nature of supplementary material to a course of instruction. With very few exceptions, the statements of the problems appear in the author's "Cours de mathématiques" (Masson, Paris, 1964). As a whole, the exercises chosen are of medium difficulty. The more difficult ones are indicated by an asterisk preceding the title of the exercise.

The subjects treated have been divided into six chapters which are devoted to simple integrals, uniform convergence and normed spaces, line integrals and multiple integrals, analytic functions, ordinary differential equations, and partial differential equations. This order is not happenstance. In each chapter the reader should be capable of using the material constituting the subject of the exercises of the preceding chapters. Although the beginning of Chapter 2 is devoted to uniform convergence, numerous exercises in the following chapters constitute a training in the techniques of continuity of series and integrals, and of differentiation and integration under the integral sign. Similarly, the residue theorem, which is the subject of the end of Chapter 4, is used in numerous exercises in Chapters 5 and 6. In Chapter 1 there are no exercises on the convergence of improper integrals. Such exercises inevitably constitute an initial problem in various exercises in Chapter 2 dealing with the uniform convergence of integrals. Orthogonal functions appear first in Chapter 2 and again in Chapter 5 in connection with problems on second-order linear differential equations. Very few topics from the calculus of variations have been treated. They appear at the end of Chapter 5 although they are sometimes more of an algebraic than a functional nature.

Here we do not propose to give exercises on probability theory. Such a subject requires special preparation. However, the relationships between probability theory and certain questions in algebra and integral calculus are so close that several exercises are worded in probability theory language. For the reader who may not be familiar with the elements of probability theory, the few words of introduction preceding the statement of the problems will enable him to make the necessary adjustment.

J. BASS

February, 1966

A Note about the French Edition

The original French edition of this volume contains all the material found here and, in addition, a chapter on linear algebra and a short chapter on probability. It corresponds very closely to the author's "Cours de mathématiques" (3rd ed., Masson, Paris, 1964) and, with only a few exceptions, the statements of the problems can be found there. Among the French texts suitable for background study are:

J. Bass, "Éléments de calcul des probabilités." Masson, Paris, 1964. An English translation, "Elements of Probability Theory," will be published by Academic Press in 1966.

A. Hocquenghem and P. Jaffard, "Mathématiques." Masson, Paris, 1964.

J. Kuntzmann, "Mathématiques de la physique et de la technique." Hermann, Paris.

L. Schwartz, "Méthodes mathématiques pour les sciences physiques." Hermann, Paris, 1961.

The rough drafts of these exercises were prepared by the assistants of l'École Nationale Superieure de l'Aéronautique between 1962 and 1964. MM. J. Azema, J. P. Bertrandias, J. Couot, M. Gatesoupe, M. Mendès-France, M. Pianko, J. Servant, and Vo Khac Khoan had the responsibility for individual chapters, and I am very grateful to them for their assistance. I also wish to thank MM. J. Dhombres, F. Dress, F. Hoffman, J-F. Mela, Pham Phu Hien, and Ph. Robba who reread the manuscripts and the proofs and the former assistants at l'École Nationale Supérieure de l'Aéronautique and l'École Nationale Superieure des Mines — MM. P. Belayche, F. Bourion, F. Germain, J. Germain, J-P. Guiraud, G. Legrand, J. Stern, and R. Vallée — to whom I am indebted for many interesting comments.

J. BASS

Contents

CHAPTER 1

Sequences, Series, and Definite Integrals

This chapter contains several exercises on the fundamental concepts of analysis. These exercises do not require extensive preparation on the part of the reader. It will be sufficient for him to know various definitions that are part of almost all curricula or special mathematical courses and certain elementary properties of the objects in question. The most frequently encountered are the following:

Distance in general. The distance between two functions. The distance between two sequences.

Norms in a vector space. The norm of a function.

Classical properties of continuous and monotonic functions.

Convergence of numerical sequences and series. Absolute convergence. Alternating series.

Definite integrals. Integrable functions of a single variable. The first and second mean-value theorems. Schwarz' inequality.

The integrals, of course, are in the sense of Riemann. The exercises on Hölder's inequality would have quite extensive sequels if we were using the Lebesgue integral. However, since we are dealing here with purely algebraic properties of the *definite* integral, only the elementary axioms come into play: The integral is a linear functional and is additive as a function of the interval of integration; integrable functions constitute an algebra; the integral of a positive function is positive; the integral of the absolute value of a function is at least as great as the absolute value of the integral of the function; if f is nonnegative and its integral is zero, then f is zero almost everywhere. With these principles, it hardly matters what kind of integral one is using.

All the definite integrals are of bounded functions defined on a compact interval. The chapter contains no exercises on "generalized" (i.e., improper) integrals, either integrals over an infinite interval or integrals of unbounded functions. Integrals of this type will appear frequently in Chapter 2 in connection with exercises on uniform convergence. Among the questions posed, one frequently encountered will be to prove the convergence of an integral, and it has not seemed advisable to isolate these exercises. We therefore refer the reader to Chapter 2. Also in Chapter 2 are exercises on the Bolzano-Weierstrass theorem, in connection with uniform convergence.

1. Examples of the distance between two functions and between two sequences

Let us give an example of a distance between two functions that is not a norm.

Problems. A. Suppose that a and b are positive numbers. Show that

$$\frac{a+b}{1+a+b} \leqslant \frac{a}{1+a} + \frac{b}{1+b}.$$

B. Let $f(x)$ be an integrable function on the interval $\alpha \leqslant x \leqslant \beta$ and set

$$L(f) = \int_\alpha^\beta \frac{|f(x)|}{1+|f(x)|}\,dx.$$

Show that

$$L(f+g) \leqslant L(f) + L(g).$$

Can we define the distance between two functions with the aid of the functional L?

C. Let u and v be any two sequences of real numbers the values of which are $u_1, \ldots, u_n, \ldots$ and $v_1, \ldots, v_n \ldots$. Let a_n be the general term of a convergent sequence of positive numbers. Set

$$L(u) = \sum_n a_n \frac{|u_n|}{1+|u_n|}, \quad L(v) = \sum_n a_n \frac{|v_n|}{1+|v_n|}.$$

Denote by $u + v$ the sequence whose general term is $u_n + v_n$. Show that

$$L(u + v) \leqslant L(u) + L(v).$$

Solutions. **A.** Since a and b are positive numbers, the inequality

$$\frac{a+b}{1+a+b} \leqslant \frac{a}{1+a} + \frac{b}{1+b} \tag{1}$$

follows from the obvious inequalities

$$\frac{a}{1+a+b} \leqslant \frac{a}{1+a}, \qquad \frac{b}{1+a+b} \leqslant \frac{b}{1+b}.$$

Equality can hold only if a or b is zero.

B. Since f is an integrable function on the interval $[\alpha, \beta]$, the functional L maps f into the positive number

$$L(f) = \int_{\alpha}^{\beta} \frac{|f(x)|}{1 + |f(x)|}\, dx \qquad (\alpha < \beta).$$

Let us show that the functional L verifies the triangle inequality

$$L(f + g) \leqslant L(f) + L(g). \tag{2}$$

First of all, we have

$$|f + g| \leqslant |f| + |g|.$$

Since the function $x/(1 + x)$ is an increasing function, we obtain

$$\frac{|f+g|}{1+|f+g|} \leqslant \frac{|f|+|g|}{1+|f|+|g|}$$

$$\leqslant \frac{|f|}{1+|f|} + \frac{|g|}{1+|g|}$$

on the basis of inequality (1). This proves inequality (2).

Let us now recall the definition of the distance between two functions: The distance $l(f, g)$ between two functions f and g is a real number that satisfies the following axioms:

(a) $l(f, g) > 0 \quad$ if $f \neq g$,

(b) $l(f, f) = 0$,

(c) $l(f, g) = l(g, f)$,

(d) $l(f, g) \leqslant l(f, h) + l(h, g) \qquad$ for arbitrary functions f, g, h.

If we agree to consider two functions f and g equal when $L(f - g) = 0$, we see that the functional L enables us to define the distance between two functions f and g. We simply set

$$l(f, g) = L(f - g).$$

This definition is valid for integrable functions defined on the interval $[\alpha, \beta]$. The functional $L(f)$ may be used to define the distance between two functions, but it is not a *norm* in the vector space of functions that are integrable on $[\alpha, \beta]$. To see this, consider the vector space of real functions. Here, if m denotes a real number, the requirement of homogeneity $L(mf) = |m| L(f)$ is not verified.

C. We have seen that

$$\frac{|u_n + v_n|}{1 + |u_n + v_n|} \leqslant \frac{|u_n|}{1 + |u_n|} + \frac{|v_n|}{1 + |v_n|}.$$

Since $|u_n|/(1 + |u_n|) \leqslant 1$, the series $\sum a_n |u_n|/(1 + |u_n|)$ converges.

If we multiply the two sides of the above inequality by a_n and sum over n, we obtain

$$L(u + v) \leqslant L(u) + L(v),$$

so that

$$L(u) = \sum a_n \frac{|u_n|}{1 + |u_n|}.$$

The functional $L(u - v)$ can be considered as the *distance between the two series* u and v. It is positive except when $u_n = v_n$ for every n, in which case it is zero. It is symmetric with respect to u and v. Finally, it satisfies the triangle inequality. It is interesting to note that this definition of distance is valid for infinite sequences *irrespective of their convergence or divergence.*

2. Extreme values of continuous functions

Problems. A. Let $f(x)$ be a continuous function that is positive everywhere in the closed interval $[a, b]$, where $a < b$. Then $f(x)$ attains a maximum value M at a point x_0 in $[a, b]$. Show that

$$M = \lim_{n \to +\infty} \left[\frac{1}{b - a} \int_a^b [f(x)]^n \, dx\right]^{1/n}.$$

B. Show that, if $f(x)$ is strictly positive, the minimum of $f(x)$ is equal to

$$\lim_{n \to -\infty} \left[\frac{1}{b-a}\int_a^b [f(x)]^n \, dx\right]^{1/n}.$$

Solutions. **A.** By hypothesis, the function f is positive and continuous on the closed interval $[a, b]$. Therefore, it attains its maximum M at least once on $[a, b]$, at some point x_0. Let us suppose first that x_0 is different from a and b. For any $\epsilon > 0$, there exists a number α less than $x_0 - a$ and $b - x_0$ such that the inequality

$$|x - x_0| < \alpha$$

implies that

$$M - \epsilon \leqslant f(x) \leqslant M.$$

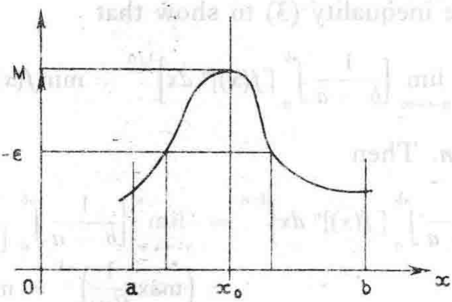

FIGURE 1

Let us denote by E the union of the two intervals $(a, x_0 - \alpha)$ and $(x_0 + \alpha, b)$. Then

$$\frac{1}{b-a}\int_a^b [f(x)]^n \, dx = \frac{1}{b-a}\int_{x_0-\alpha}^{x_0+\alpha} [f(x)]^n \, dx + \frac{1}{b-a}\int_E [f(x)]^n \, dx.$$

Consequently,

$$\frac{1}{b-a}\int_a^b [f(x)]^n \, dx \geqslant \frac{1}{b-a} 2\alpha(M-\epsilon)^n. \tag{1}$$

On the other hand, the inequality $f(x) \leqslant M$ implies that

$$\frac{1}{b-a}\int_a^b [f(x)]^n \, dx \leqslant M^n,$$

so that

$$\left(\frac{2\alpha}{b-a}\right)^{1/n} (M - \epsilon) \leqslant \left[\frac{1}{b-a}\int_a^b [f(x)]^n \, dx\right]^{1/n} \leqslant M. \tag{2}$$

As n increases without bound, the factor $[2\alpha/(b-a)]^{1/n}$ tends to 1. Therefore, there exists a number N_ϵ such that the inequality $n > N_\epsilon$ implies the double inequality

$$M - 2\epsilon \leqslant \left[\frac{1}{b-a}\int_a^b [f(x)]^n\,dx\right]^{1/n} \leqslant M.$$

This double inequality means that

$$\lim_{n\to+\infty}\left[\frac{1}{b-a}\int_a^b [f(x)]^n\,dx\right]^{1/n} = M. \tag{3}$$

Remark. In the above, the number α was chosen sufficiently small that the interval $(x_0-\alpha, x_0+\alpha)$ was contained in the interval (a, b). In the event that x_0 is, let us say, the point a, it will be sufficient to replace the interval $(x_0-\alpha, x_0+\alpha)$ with $(a, x_0+\alpha)$.

B. Let us use inequality (3) to show that

$$\lim_{n\to-\infty}\left[\frac{1}{b-a}\int_a^b [f(x)]^n\,dx\right]^{1/n} = \min f(x). \tag{4}$$

We set $n' = -n$. Then

$$\lim_{n\to-\infty}\left[\frac{1}{b-a}\int_a^b [f(x)]^n\,dx\right]^{1/n} = \lim_{n'\to+\infty}\left[\frac{1}{b-a}\int_a^b \frac{1}{[f(x)]^{n'}}\,dx\right]^{-1/n'}$$
$$= \left(\max\frac{1}{f(x)}\right)^{-1} = \min f(x).$$

The function $1/f(x)$ verifies the conditions of problem A, since $f(x)$ does not vanish on the closed interval $[a, b]$ and has a positive minimum in that interval.

C. These formulas express the extreme values of a continuous function on an interval in terms of a limit without requiring that the function be differentiable. Therefore, this procedure is more general (although less practical) than that of finding the points at which the derivative of $f(x)$ (which is also the result of a limiting process) vanishes.

Example. $f(x) = 1 - x$ for $0 \leqslant x \leqslant 1$.
The maximum of $f(x)$ is given by

$$\lim_{n\to\infty}\left[\int_0^1 (1-x)^n\,dx\right]^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n+1}\right)^{1/n} = 1.$$

This maximum appears at the point [well defined since $f(x)$ is continuous] at which $f(x) = 1$. This is the point $x = 0$.