CALCULUS

MUNROE

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CALCULUS

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Preface

This book presents a three-semester sequence in calculus, beginning at the beginning and culminating in Stokes' Theorem and an introduction to differential forms.

The treatment is definitely not a rigorous one. We make no attempt to describe the real number system or to develop the theory of limits. Indeed, our basic philosophy on "limits for the beginning calculus student" is incorporated in Section 5-5. In a nutshell, it is that for continuous functions limits can be evaluated by substitution, that calculus courses deal with elementary functions, and that these are continuous except at obvious points. With this attitude toward the basic background, we obviously do not prove many theorems in this book. However, we are convinced that the student will swallow only so much simply "because the book says so." Therefore, we present as many plausibility arguments as we can, billing them quite candidly as just that.

The major innovation in this book is the treatment of the differential. Before World War II people blithely wrote calculus books giving a nonsensical "definition" of differential and then using it as though it meant something. There was enough hue and cry about this so that later authors backed away from the differential to such an extent that in some books the notation dy/dx for a derivative does not appear. In a recent meeting of the Mathematical Association of America a distinguished panel of applied mathematicians deplored this trend and pleaded, "Give us back the Leibnitz notation in elementary calculus." The machinery to do this and more was developed by the differential geometers before World War II, but many still regard this as "advanced" mathematics and therefore classified information so far as freshmen are concerned. The student's ability to use calculus is enhanced manyfold if he has confidence in the technique of pushing differentials around to derive new formulas. In the light of modern knowledge this procedure is completely justified and we feel strongly that to continue to suppress it at the freshman level is like taking a

Victorian attitude toward sex education. It simply cannot be justified in the latter half of the Twentieth Century.

The modern definition of a differential is rather sophisticated and therefore we try to present it in stages. In the first three sections of Chapter 3 we give, very informally, enough of the theory to get the program off the ground. We tighten up on it a little in Section 5-9. In Section 14-4 we present the multidimensional version, and we finally tell the whole story (still without proving any tough theorems) in Section 14-6. In the final analysis, what the student needs is not the details of the theory but the conviction that there is a logically sound theory in which dx comes from x and dy comes (quite independently) from y. Then, the working mechanism is the *theorem* that a non-linear relation $y=f(x)$ generates on the tangent spaces a linear relation $dy=f'(x)\,dx$. And we are off and running with the Leibnitz notation for the derivative!

The Committee on the Undergraduate Program in Mathematics has for some time been advocating the sandwiching of linear algebra into the calculus sequence. This has produced such ludicrous phenomena as three-semester calculus texts with a linear algebra chapter at the end! Actually, C.U.P.M. was more explicit than this. They recommended teaching linear algebra in the middle and using it in multidimensional calculus. In our opinion, however, C.U.P.M. did not really face the issue. The real magic of linear algebra in multidimensional calculus stems from the fact that *if the differential is properly defined,* then linear algebra on the tangent bundle yields meaningful results on the underlying manifold. Once we have developed the appropriate background, we make extensive use of linear algebra from Chapter 14 on. The necessary introduction to the subject is in Chapter 13; but, if desired, a separate, more extensive, course in linear algebra from another text may be substituted for Chapter 13.

One word of caution about a separate linear algebra course: Be sure it gives adequate coverage of change of basis because this is the name of the game in multidimensional calculus. A differentiable coordinate transformation on a manifold generates a linear change of basis on each tangent space. To this end, note that if you define a vector as an n-tuple of numbers, you have had it. Each different basis associates a given vector with a different n-tuple of numbers.

Experience with preliminary editions of this book shows that Chapters 1 to 12 constitute a reasonable first-year course (probably 8 semester hours). To cover the remainder in one semester may require a little editing. Specific suggestions for cutting this material are included in the Instructor's Manual. We do want to enter a plea, however, for the preservation of Chapter 19 pretty much *in toto.* In a sense, the entire book is built toward this as a climax. No student can maintain steady enthusiasm for calculus, but most of them come away with a good taste in their mouths after seeing the generalized Stokes' Theorem spawn specializations and applications as it does in this final chapter.

M. E. M.

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One

A Preview

1-1. Introduction

Archimedes was killed in 212 B.C. during the Roman capture of Syracuse. He left instructions that his epitaph should consist of a drawing of a sphere and a cylinder. He felt that to have found formulas for the area and volume of these figures was the crowning achievement of his long scientific career. This was a strangely prophetic evaluation of Archimedes' accomplishments, for his work on areas and volumes was essentially integral calculus.

The reason calculus was not born in the third century B.C. was that Archimedes developed only one of the two basic ideas involved in the subject. For 1900 years after Archimedes, very little more was accomplished in the development of calculus. Then, Newton (1642–1727) and Leibnitz (1646–1716), working independently, discovered that the study of velocities of moving particles is intimately connected with the study of areas and volumes. The study of velocities is an example of differential calculus, and the connection between this and integral calculus (the so-called Fundamental Theorem of Calculus) allowed the subject to flourish and blossom into the many-sided discipline that it has become since the day of Newton and Leibnitz.

It is, of course, an oversimplification to say that Newton and Leibnitz "invented" calculus. They depended heavily on their predecessors, and a great number of essential features have been added to the subject since their time. Roughly speaking, there have been two main lines of development in calculus since 1700: formal developments—the discovery of new formulas and techniques, and basic developments—the critical study of the underlying ideas and principles on which calculus is based. Though both these lines of development have proceeded (and still are proceeding) simultaneously, the eighteenth century is frequently thought of as the golden age of formal development in calculus, while the nineteenth century is regarded as the most important era of basic development. It should be noted, however, that the twentieth century has seen a significant basic development in calculus.

An oversimplified but suggestive summary of this history would be to say that in the eighteenth century they got the answers; in the nineteenth, a logical analysis of the intermediate steps; and in the twentieth, a clear idea of the starting point. In terminology suggested by this generalization, the present book might be classed as eighteenth century calculus with the twentieth century improvements. No attempt will be made to fill in the nineteenth century contributions, because experience has shown that this is the difficult part of calculus for the beginner. Thus, for the most part, arguments will be intuitive rather than logical, but the state of modern knowledge will be exploited to the fullest in the formulation of basic ideas.

1-2. Summation

Let $a_1, a_2, \ldots, a_n$ be an ordered set of n numbers. The sum of these,

$$a_1 + a_2 + \cdots + a_n,$$

is often denoted by

$$\sum_{i=1}^{n} a_i. \tag{1}$$

The symbol $\sum$ is called a *summation sign*, and the symbol i in (1) is called the *summation index*. Given the same set of numbers as above, other sums may be formed; for example,

$$\sum_{i=3}^{k} a_i = a_3 + a_4 + a_5 + \cdots + a_k \qquad (3 \leq k \leq n),$$

$$\sum_{i=1}^{k} a_{2i} = a_2 + a_4 + a_6 + \cdots + a_{2k} \qquad \left(1 \leq k \leq \frac{n}{2}\right).$$

More generally, if m, n, j, k are integers with $m \leq j \leq k \leq n$, and if $a_m, a_{m+1}, \ldots, a_n$ is an ordered set of numbers, define

$$\sum_{i=j}^{k} a_i = a_j + a_{j+1} + a_{j+2} + \cdots + a_k.$$

Here j and k are called the *lower* and *upper limits of summation*, respectively. Informally, the summation sign means: "Assign to the summation index successive integer values from the lower to the upper limit of summation, inclusive. For each of these values of the summation index, evaluate the expression behind the summation sign, and compute the sum of all these results."

More precisely, the summation symbol may be defined inductively as follows:

$$\sum_{i=k}^{k} a_i = a_k, \qquad \sum_{i=k}^{n} a_i = a_n + \sum_{i=k}^{n-1} a_i.$$

Note that the result does not depend on the summation index; therefore the letter used for this index is immaterial; that is, each of the symbols,

$$\sum_{m=1}^{n} a_m, \qquad \sum_{k=1}^{n} a_k, \qquad \sum_{j=1}^{n} a_j,$$

means the same thing as (1).

EXAMPLES

1. $\sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91.$

2. Prove by mathematical induction that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Recall that there are two steps in an inductive proof: (i) Verify the formula for $n = 1$. (ii) Show that if it holds for n, then it holds for $n + 1$. In this case note that

$$\sum_{i=1}^{1} i^2 = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6};$$

so the result holds for $n = 1$. Assuming that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

add $(n+1)^2$ to each side of the equation:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \left(\frac{n+1}{6}\right)(2n^2 + n + 6n + 6) = \left(\frac{n+1}{6}\right)(n+2)(2n+3)$$

$$= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6},$$

and the induction is complete. The reason for adding $(n+1)^2$ to each side of (2) is that this yields the correct form for the $(n+1)^{\text{st}}$ case on the left side of the equation. The proof is then completed by routine computation to show that it also yields the correct form on the right.

Informally, the line of argument here is that the result is correct for $n = 1$ and that, given any value for n for which it is correct, the result also holds for the next positive integer. From this it is inferred that the result holds for every positive integer value of n. This seems reasonable, but in the final analysis the validity of this argument rests not on a principle of logic but on a property of the set of positive integers. One of the definitive properties of the set of positive integers is that it is exhausted by the finite induction process illustrated here.

EXERCISES

1. Evaluate each of the following:

a. $\sum_{i=3}^{7} 2i$

b. $\sum_{i=3}^{7} i^2$

c. $\sum_{i=1}^{4} \frac{1}{i}$

d. $\sum_{i=2}^{4} \frac{1}{i^2}$

e. $\sum_{i=1}^{5} (2i+3)$

f. $\sum_{i=1}^{4} (i^2-1)$

g. $\sum_{i=5}^{8} (2i+1)^2$

h. $\sum_{i=2}^{5} \frac{1+i}{1-i}$

i. $\sum_{i=3}^{7} \frac{2i}{2i+1}$

j. $\sum_{i=3}^{6} \frac{i}{1+i^2}$

2. Write each of the following with a summation sign.

a. $1+3+5+7+9+11$
b. $1+4+9+16+25+36+49+64$
c. $1+9+25+49+81+121+169$
d. $1+3+5+\cdots+(2n+1)$
e. $2+4+6+\cdots+(2n+2)$
f. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)$
g. $\sqrt{2}+\sqrt{5}+\sqrt{10}+\cdots+\sqrt{1+n^2}$
h. $c_1^2(c_1-c_0)+c_2^2(c_2-c_1)+\cdots+c_n^2(c_n-c_{n-1})$
i. $\sqrt{1-c_1}\,(c_1-c_0)+\sqrt{1-c_2}\,(c_2-c_1)+\cdots+\sqrt{1-c_n}\,(c_n-c_{n-1})$

3. Prove each of the following by mathematical induction.

a. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

b. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

c. $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

d. $(a+b)^n = \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!}\, a^{n-i}b^i$

4. a. Compute

$$\sum_{i=3}^{7} (i^2+2i).$$

Compare Exercises 1a and 1b above.

b. Prove that

$$\sum_{i=k}^{n} (a_i + b_i) = \sum_{i=k}^{n} a_i + \sum_{i=k}^{n} b_i.$$

What properties of the operation of addition are needed in this proof?

c. Apply the result of part b to Exercise 1e. State the meaning of

$$\sum_{i=1}^{5} 3.$$

d. If c is a number, give the value of

$$\sum_{i=k}^{n} c.$$

5. a. Show that

$$\sum_{i=3}^{n} a_i = \sum_{i=2}^{n-1} a_{i+1} = \sum_{i=4}^{n+1} a_{i-1} = \sum_{i=0}^{n-3} a_{n-i}.$$

b. Write three other forms yielding the same sum.

1-3. Functions

One great advantage that Newton and Leibnitz had over Archimedes was access to the work of Descartes (1596–1650) and others on analytic geometry. This work might be characterized as the systematic study of equations and their graphs. The detailed study of analytic geometry can be dispensed with for the present, but in order to understand calculus it is essential to have a clear picture of the fundamental ideas on which analytic geometry is based. This section and the next two will present a modern revised version of Descartes' basic discoveries.

In many different connections one sees tabulations of numbers in two parallel columns. A simple example:

2	−1
−1	4
0	5
3	−2
−5	−1

(1)

As a general rule, the columns are labeled to indicate that the entries represent measurements of some sort. Essentially, this introduces additional concepts (see Section 1-5); so the labels have been omitted here in an effort to distill one basic idea for discussion in the present section.

If one reads across rather than down, the table (1) appears to consist of five ordered pairs of numbers. For example, the first row in (1) reads

$$2 \quad -1.$$

To say that this is an ordered pair of numbers is to distinguish it from

$$-1 \quad 2,$$

which consists of the same two numbers in the reverse order. The notation (a, b) is commonly used for the ordered pair whose first entry is a and whose second entry is b. In this notation (1) would be written

$$(2, -1), \qquad (-1, 4), \qquad (0, 5), \qquad (3, -2), \qquad (-5, -1).$$

Now, a *function* is defined as *a set of ordered pairs of numbers no two of which have the same first entry.*

Note that (1) is an example of a function. The first and last pairs in (1) have the same second entry, but this is immaterial. Only duplications among the first entries are ruled out in the definition of a function.

To turn to other familiar examples, note that for purposes of numerical computation, logarithmic and trigonometric functions are finite sets of ordered pairs displayed in a book of tables.

The *domain* of a function is the set of all first entries in its ordered pairs; the *range* of a function is the set of all its second entries. For the function (1) the domain is displayed in the first column, and the range in the second column.

The order in which the ordered pairs of a function are listed is of no significance. That is, by definition,

−5	−1
−1	4
0	5
2	−1
3	−2

(1′)

is the same function as (1). Often it is convenient to arrange a function as in (1′), putting the numbers of the domain in increasing order. However, sometimes another arrangement is more convenient; and if this is the case, the rearrangement is quite permissible.

Frequently, a single letter is used to denote a particular function. The ones in most common use are f, g, F, G, ϕ, ψ; though occasionally others are introduced as needed. If f is a function and a is a number in its domain, then the symbol

$$f(a)$$

is used to denote the entry in the range corresponding to a. The symbol $f(a)$ is read, "f of a," and is called the *value of f at a*. Given a, the operation of getting $f(a)$ is called *application* of f to a. If, for example, f denotes the function displayed in (1), then

$$f(2) = -1, \qquad f(-1) = 4, \qquad \text{etc.}$$

If a, b and c are numbers, then $a(b + c)$ means, "a times the number $b + c$." Parentheses will still be used in this way, but the function value symbol introduces a new use for parentheses that generally has nothing to do with multiplication. This creates no confusion provided it is borne in mind that parentheses signal application of a function if and only if two conditions prevail: (i) The symbol before the parentheses is one for a function. (ii) The symbol inside the parentheses is one for a number in the domain of this function.

The next step is to define the sum of two functions. Note that if f and g are functions, $f + g$ is not intrinsically defined. Other definitions could be devised, but the following has proved to be useful and is generally adopted. If f and g are functions, $f + g$ is the function consisting of all ordered pairs of numbers $(a, b + c)$ where (a, b) is in f and (a, c) is in g. Informally, pair off (as far as possible) equal entries in the two domains, and add corresponding entries in the ranges. Examples:

f		g		$f+g$	
−5	−1	−5	−1	−5	−2
−1	4	−1	3	−1	7
0	5	0	−2	0	3
2	−1				
3	−2	3	0	3	−2
		4	4		

Subtraction, multiplication and division are defined in a similar manner. The student should formulate precise definitions. Examples:

f		g		$f-g$		fg		f/g	
−5	−1	−5	−1	−5	0	−5	1	−5	1
−1	4	−1	3	−1	1	−1	12	−1	4/3
0	5	0	−2	0	7	0	−10	0	−5/2
2	−1								
3	−2	3	0	3	−2	3	0		
		4	4						

Note that division by zero is not defined; where it is indicated, that ordered pair is deleted from f/g.

Multiplication of functions introduces in a natural way the positive integer powers of a function. That is, $f^2 = ff$; $f^3 = fff$. In general,

$$f^n = f^{n-1}f.$$

The notion of a fractional exponent requires a more elaborate discussion for a careful definition. Such a discussion will appear in Chapter Four, but by way of expanding the list of examples, fractional exponents will appear in this chapter. Briefly,

$$f^{1/n}$$