ADVANCED TECHNICAL MATHEMATICS

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North American Rockwell Corporation
and
Ohio Technical College

CHARLES E. MERRILL PUBLISHING COMPANY
A Bell & Howell Company
Columbus, Ohio

Library of Congress Catalog Card Number:
69-19744
Standard Book Number: 675-09509-3

1 2 3 4 5 6 7 8 9 10-74 73 72 71 70

Printed in the United States of America

To my sons,
Kevin and David

Preface

This text is designed specifically for students who require the *essential definitions* of certain "advanced" technical mathematical procedures as well as the opportunity to *practice* these procedures. Hence, the author has deliberately avoided the maze of symbology, derivation, and rigorous proofs that traditionally fill most advanced mathematics texts. Algebra and trigonometry are the only prerequisites for the understanding of the material presented. The reader with background in analytic geometry will have an advantage, but adequate material on this subject is provided in the first few chapters.

Chapters 1 and 2 constitute a review of fundamental algebraic concepts along with analytical relationships for experimentally determined data. Also introduced is the basic analytic geometry for linear, parabolic, and exponential functions. Chapter 3 covers vectors and vector operations, along with an introduction to the theory of complex numbers and variables.

Calculus, both differential and integral, is presented in Chapters 4 and 5, followed by hyperbolic functions, infinite series, and gamma and beta functions in Chapter 6. Although hyperbolic functions and infinite series could be treated in a strictly algebraic fashion, it has been the author's experience that by presenting them in terms of the calculus, the coefficients of the Maclaurin, Taylor, and Fourier series are arrived upon most simply.

Almost all investigations of mechanical, electrical, electronic, and associated system behavior are limited to steady state (equilibrium) conditions unless the differential equations that describe their behavior are developed and evaluated. Such equations are presented in Chapter 7 along with the common techniques for obtaining general and particular solutions.

Chapter 7 introduces the Laplace transformation: the calculus "operator" that converts differentiation into multiplication and intergration into division, thereby transforming the solution of many differential equations into a process of algebraic manipulation and appendix reference. A comprehensive set of reference tables, including logarithms, exponentials, dimensions, conversion factors, trigonometric functions, derivatives, integrals, hyperbolic functions, gamma functions, selected definite integrals, and Laplace transforms provides all the reference material necessary to implement the concepts presented in this text.

Upon completion of this volume, the student should feel qualified to undertake the majority of the more intricate and actual assignments of industry.

The author wishes to express his appreciation to Samuel West of the University of Texas at Arlington for his review of the original manuscript, and to Mrs. Ann Mirels of the Merrill Staff for her time and efforts expended in designing and editing this text.

Louis H. Lenert

January, 1970

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1

Basic Concepts

Mathematics is a study of the use of numbers and symbols to express relationships between various quantities. To the theoretical mathematician, the symbolic relationships, no matter how abstract, are the most important. Even the applied mathematician (and, certainly, scientists, engineers, and technicians *are* applied mathematicians) finds that abstract symbology is quite useful in defining, organizing, and analyzing the problems which confront him. Furthermore, the symbology often simplifies the solution of problems, since it permits the easy identification of similar groupings and expressions. For this reason, it is recommended that the reader become accustomed to manipulating the symbols, resorting to the substitution of numbers for these symbols only as a very last step in the solution process.

The discipline associated with developing the mathematical formulation of a problem forces an equal discipline upon "thinking" about the problem. Much confusion can be avoided by adopting such a discipline. The procedure[1] for solving scientific and engineering problems may be divided into four steps:

[1]C. M. Haberman, *Engineering Systems Analysis* (Columbus, Ohio: Charles E. Merrill Books, Inc., 1965), p. 1.

1. a concise definition of the problem, including all important factors, assumptions, and limitations;
2. the formulation of a physical model (a "picture" of the problem may help), including a listing of all associated scientific and engineering principles;
3. the formulation of a complete set of mathematical equations, including all dimensions and all possible simplifications of these equations; and
4. the solution of the set of equations (utilizing analog and digital computers if necessary), with particular attention paid to accuracy.

Such a procedure is perfectly general and will be followed throughout this book. In most examples and problems, the problem definition and physical model will be presented—the formulation and solution of the appropriate set of equations will comprise the "work" to be accomplished. This approach is taken because the book is devoted to the formulation of new types of mathematical relationships and the techniques for manipulating them to a solution, rather than to the development of scientific and engineering principles. However, because of the *applied* nature of the book, all of the mathematical concepts will be exercised directly in the solution of scientific and engineering problems. In this respect, it is not assumed that the reader has any particular technical specialty. The reader may be surprised at how universal most mathematical techniques are.

1.1. Variables and Constants

Mathematical quantities lie in one of two general categories: *variables* and *constants*. Both quantities must ultimately be assigned numerical values for them to be of any practical use. It is the nature of the assigned values that determines in which category the quantity belongs. In a given discussion, if the numerical value of a quantity does not change, the quantity is known as a constant. If a quantity can take on different values, *either any value without limit*, *or any value between specific limits*, it is known as a variable.

The expression:

$$E_k = \tfrac{1}{2}mv^2$$

relates kinetic energy (E_k) to the mass (m) and velocity (v) of an object. In any single situation, the mass of an object may be fixed (constant). The velocity, however, may vary—almost without limit. Because of the equality of the expression, the kinetic energy must also be a variable (E_k changes value as the value of v is changed). In another situation, or problem, the object may be changed—changing the value of m. However, for the "duration" of this problem, the mass may again remain constant, and v and E_k may again vary in almost any manner.

This is not to imply that mass (or any other quantity) is always a constant in every expression. The expression:

$$E = mc^2$$

relates the amount of energy (E) released during a nuclear reaction to the amount of mass lost (m) and to the velocity of light in free space (c). For situations involving this expression, c is always a constant (approximately 3×10^8 meters/second). The mass and energy are variables.

It should be apparent, from this discussion, that (in general) quantities and symbols, themselves, do not defiine variables and constants. The definition depends *only on the manner in which the quantity is treated in a particular situation.* For purely mathematical discussions, variables are usually represented by letters near the end of the alphabet (r, s, t, u, v, w, x, y, and z), while the letters near the front of the alphabet (a, b, c, d, and e) are reserved for a representation of constants. For example:

$$y = ax^2 + bx + c$$

is a general mathematical expression where y and x are variables; a, b, and c are constants. For the set of equations:

$$y = 3x^2 + 2x + 4$$
$$y = 5x^2 - 4x + 1$$

a, b, and c take on different values (3, 2, 4 and 5, -4, 1 respectively). However, for each equation, a, b, and c assume fixed values. This particular use of certain

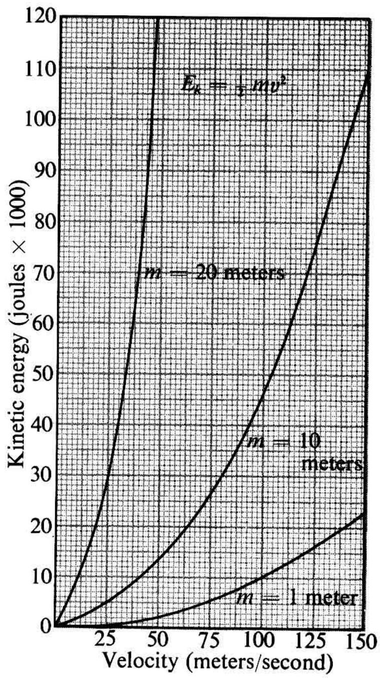

FIGURE 1-1. Parametric Presentation of an Equation

letters of the alphabet for variables and constants is one of convenience, not one of necessity. In practical problems, the letter for each quantity is (most frequently) assciated with the quantity itself (p for power, v for voltage or velocity, r for resistance, etc.). Any of these may be constants or variables, depending on the situation.

Constants which take on different numerical values from one situation to another are *conditional constants* or, more technically, *parameters* of an equation. The mass (m) in the kinetic energy equation described previously is such a parameter. Figure 1–1 illustrates plots of the kinetic energy equation for different values of the parameter. Many of the so-called "physical constants" in science and engineering are conditional. This is true of the acceleration due to gravity ($g =$ 32.2 feet/second^2). This value applies only at sea level, on the planet, earth. Other constants, which do not change value under any condition, are *absolute*. The ratio between the area of a circle and the square of the radius ($\pi = 3.1415 \cdots$) and the base of the natural logarithm system ($e = 2.718 \cdots$) are absolute constants.

Problems

Identify the variables, conditional constants, and absolute constants in the following expressions.

1. The force of attraction (F) between the earth, having mass (m_1), and the moon, having mass (m_2), is related to the distance (d) between the two:

$$F = G\frac{m_1 m_2}{d^2}$$

where $G = 6.67 \times 10^{-11}$ newton•meter^2/kilogram^2.

2. The force of attraction (F) between the nucleus of a helium atom and the electrons (regardless of their orbital position) is given by:

$$F = \frac{Q_e^2 Z}{(4\pi\epsilon_0)r^2}$$

where Q_e is the charge of an electron (1.6×10^{-19} coulomb), Z is the atomic number (2, for helium), ϵ_0 is the permittivity of free space (8.85×10^{-12} coulomb^2/newton•meter^2), and r is the radius of any electron orbit.

3. The theory of relativity states that the mass (m) of any object traveling at velocity (v) is given by:

$$m = m_o \frac{1}{\sqrt{1 - (v/c)^2}}$$

where m_o is the normal "rest" mass of the object and c is the velocity of light in free space (3×10^8 meters/second).

4. A particular amplifier is designed to have a gain (A_o). Later, it is decided that a negative feedback path will be added to the amplifier. The gain with feedback is expressed by:

$$A_f = \frac{A_o}{1 + \beta A_o}$$

where β expresses the amount of feedback that is applied.

5. During a silver-plating process, the mass (M) of silver that is deposited on an object by a specific current (I) is given by:

$$M = \frac{IAt}{N_A Q_e}$$

where A is the atomic mass (107, for silver), N_A is known as Avogadro's number (6.02×10^{26} molecules/kilomole), Q_e is the charge of an electron (1.6×10^{-19} coulomb), and t is the amount of time that the current is applied.

6. A set of batteries, each having internal resistance (r_i) and producing a terminal voltage (V) is used to form a power pack. Within the power pack, the batteries are arranged so that there are a particular number (m) of batteries placed in series, and a particular number (n) of rows of these series-connected batteries in parallel. This power pack may be connected to any circuit, described by resistance (R). The amount of current that will flow in the circuit is given by:

$$I = \frac{mV}{(m/n)r_i + R}$$

Plot the following functions over the range of values indicated for the variables and for each of the parameters.

7. The electromagnetic radiation from a surface of area (A) at temperature (T) is given by:

$$P = \sigma A T^4 \text{ (watts)}$$

where $\sigma = (5.7 \times 10^{-8}$ watt/meter$^2 \cdot {}^\circ$K^4). Plot P along the vertical axis versus A (where A ranges from 1 meter^2 to 30 meters^2) along the horizontal axis, for values of $T = 10^\circ$K; 100°K; 300°K; 1000°K.

8. The thermal voltage associated with heating two dissimilar materials is given by

$$V_T = k(T - T_r)[T_n - \tfrac{1}{2}(T + T_r)] \text{ (volts)}$$

where T is the temperature of one material and T_r is a "reference" temperature (held constant) for the other material. T_n is the temperature at which the maximum voltage occurs and k is a thermal coefficient. For $k = 3.2 \times 10^{-8}$ volt/$^\circ$K^2 and $T_n = 2320$ $^\circ$K, plot V along the vertical axis versus T (where T ranges from 1000°K to 3000°K) along the horizontal axis, for values of $T_r = 0^\circ$K; 100°K; 300°K; 1000°K.

1.2. Functions

One variable (perhaps y) is said to be a function of another variable (perhaps x) if y takes on one or more values for every value of x. If:

$$y = 6x + 2$$

y is a function of x, and the function of x is the expression $6x + 2$. In this expression, every numerical value of x produces a numerical value for y.

If a particular function is to be used many times in a mathematical derivation, or in a particular discussion, it is convenient to use a symbolic *functional notation* rather than to continually write the complete function. For example:

$$y = f(x)$$

means that y is equal to the f-function of x. If:

$$f(x) = 6x + 2$$

y is defined by the same expression presented previously. However, it is now possible to manipulate the expression $f(x)$, rather than the expression $6x + 2$. Notice that the notation $f(x)$ has simply replaced the variable y. The reader may question the logic of such a replacement—why not just use the variable y throughout the discussion or derivation? The notation $f(x)$ is preferred because *y may be a function of any variable*! Writing $f(x)$ continually reminds the user that the variable x is the one that must be dealt with in solving the problem.

The selection of the f-function of x is as arbitrary as the selection of symbols for constants and variables, described previously. It would be just as appropriate to use the g-function of x: $g(x)$; or the F-function of x: $F(x)$; or any other convenient symbology. The functional notation must be consistent, however. In a given discussion or derivation, if:

$$f(x) = 3x^2 + 2x + 4$$

the same notation cannot be used to represent a second function. A second functional notation must be used:

$$g(x) = 5x^2 - 4x + 1$$

With this distinction, it is possible to refer to $f(x)$ and $g(x)$—and even to perform mathematical operations with $f(x)$ and $g(x)$—without confusion.

Frequently, the same notation is maintained for separate functions and the distinction is made by subscripting the functional symbol. In a discussion of circular motion, for example, the kinetic energy of an object might be denoted:

$$f_1(v) = \tfrac{1}{2}mv^2$$

and the acceleration of the object might be denoted:

$$f_2(v) = \frac{v^2}{r}$$

(where r is the radius of the circle) in order to maintain a distinction between the two f-functions.

As the mathematical expressions being represented become more complicated, the functional notation becomes more useful. This is especially true when a large number of variables are involved. The frequency of oscillation for a particular type electronic circuit is given by:

$$f = \frac{1}{2\pi\sqrt{L[C_1C_2/(C_1 + C_2)]}}$$

where L (an inductance) and C_1 and C_2 (capacitances) are variables. Rather than repeating such an expression throughout a discussion or a design derivation, it is more convenient to use a functional notation, such as:

$$f = h(L, C_1, C_2)$$

where the h-function is an arbitrary selection. Notice that constants (2 and π, in this case) are not used in the notation.

The functional notation is also used *when the mathematical relationship between variables is not known.* For example, the electronic circuit mentioned in the preceding paragraph might be built without any knowledge of the frequency relationship. However, experiments with the circuit will indicate that as L, C_1, or C_2 are changed, the frequency of oscillation changes. In this case:

$$f = h(L, C_1, C_2)$$

is the *only* mathematical relationship that can be written until sufficient experiments are performed, or until a circuit analysis is performed, to provide the actual expression. Such a situation exists in many scientific and engineering investigations.

The functions that have been described thus far are *algebraic functions*, because all of the operations involved are algebraic operations (addition, subtraction, multiplication, raising to a power, etc.). Functions may also be *trigonometric*:

$$F(t) = \sin \omega t$$
$$g(t) = \cos(\omega t + \phi)$$

or *logarithmic*:

$$f_1(u) = \log_{10}(u + 1)$$
$$f_2(u) = \ln(u/2)$$

or *exponential*:

$$H(x) = e^{-3x}$$

or combinations of two or more of these types.

In order to indicate that the value of a function is to be obtained for a particular value of the functional variable, this value replaces the variable in the functional notation. For example, if the value of the function:

$$g(x) = 5x^2 - 4x + 1$$

is to be obtained for $x = 2$, this is indicated by:

$$g(2) = 13^*$$

Likewise:

$$g(-1) = 10$$

$^*5(2)^2 - 4(2) + 1 = 13$

For multiple-variable functions, each variable must be assigned a value:

$$h(u, v) = 3u - 4v$$
$$h(2, -3) = 18$$

where $h(2, -3)$ indicates that $u = 2$ and $v = -3$.

Problems

Write out the complete interpretation of the following functional statements.

1. $v = f(i)$
2. $E_k = G(v)$
3. $A = g(g_m, z_m)$
4. $y = u(x, z)$
5. $f = H(R, C_T)$
6. $f = h(R_1, R_g, C_C)$

Write the indicated functional statements for the following mathematical expressions.

7. The g-function for: $\lambda = 3645.6[n^2/(n^2 - 4)]$
8. The u-function for: $E = (1.294 \times 10^{-4})T$
9. The ϕ-function for: $y = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
10. The K-function for: $V = V_s(1 + m \cos 2\pi ft)$ where m is a constant
11. The D-function for: $A = 20 \log(V_o/V_i)$
12. The ψ-function for: $y = z + e^{-4\pi x}$
13. The f-functions for: $y = 3x + 2$
    $z = 2x^2 - x + 5$
14. The G-functions for: $T_R = 2.51\ RC_T$
    $T'_R = 2.2\ RC_T$
    $f_H = 1/2\pi RC_T$
15. The h-function indicating that the gain of an amplifier with feedback (A_f) is related to the gain without feedback (A_o) and the feedback factor (β).
16. The f-function indicating that the force of attraction (F) between planetary objects is related to the masses of the object (m_1 and m_2) and the distance (d) between the objects.

For Probs. 7 through 14, indicate whether the functions are algebraic, trigonometric, logarithmic, exponential, or combinations of these types.

Evaluate the following functions, as indicated.

17. $f(x) = 7x^2 + 3x - 10$ Find: $f(0)$; $f(2)$
18. $g(z) = z^2 - 4$ Find: $g(-2)$; $g(6)$
19. $f_1(u) = \log(u + 1)$ Find: $f_1(0)$; $f_1(50)$
20. $H(x) = e^{-3x}$ Find: $H(0)$; $H(-1)$; $H(2)$
21. $h(L, C_1, C_2) = \dfrac{1}{2\pi\sqrt{L[C_1C_2/(C_1 + C_2)]}}$
    Find: $h[(3 \times 10^{-3}), (2 \times 10^{-6}), (2 \times 10^{-6})]$